LEONHARDI EULERI

OPERA OMNIA

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ADOLF KRAZER PAUL STÄCKEL

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LIPSIAE ET BEROLINI
TYPIS ET IN AEDIBUS B. G. TEUBNERI
MCMXX

LEONHARDI EULERI

COMMENTATIONES ANALYTICAE

AD THEORIAM INTEGRALIUM PERTINENTES

VOLUMEN SECUNDUM

EDIDERUNT

AUGUST GUTZMER ET ALEXANDER LIAPOUNOFF

LIPSIAE ET BEROLINI
TYPIS ET IN AEDIBUS B. G. TEUBNERI
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VORWORT

Die Abhandlungen 475—594 des vorliegenden Bandes sind von A. GUTZMER, dem Herausgeber des Bandes I_{17}, bearbeitet worden, der auch, zusammen mit dem Redaktionskomitee, die Korrektur und die Revision der zugehörigen Druckbogen besorgt hat.

Die Bearbeitung des Restes, nämlich der Abhandlungen 606—653, sowie des ganzen folgenden Bandes I_{19} war von A. LIAPOUNOFF übernommen worden. Er hat diesen seinen Anteil an der Eulerausgabe dem Redaktionskomitee auch rechtzeitig eingeliefert; es war ihm aber leider nicht beschieden, sich an der Drucklegung zu beteiligen. Nachdem die Kriegsverhältnisse schon vor Jahren jede Verbindung mit ihm abgeschnitten und den Druck des fertig gesetzten, aber erst zur Hälfte korrigierten Bandes I_{18} aufs ungewisse hinausgeschoben hatten, ist er am 3. November 1918 gestorben. Die Schweizerische Naturforschende Gesellschaft wird seiner Mitarbeit und seines lebhaften Interesses für die Eulerausgabe stets dankbar gedenken.

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Zürich, im Juli 1920.

FERDINAND RUDIO.

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475, 499, 500, 521, 539, 572, 587, 588, 589, 594, 606, 620, 621, 629, 630, 635, 640, 651, 653

SPECULATIONES ANALYTICAE

Commentatio 475 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 20 (1775), 1776, p. 59—79
Summarium ibidem p. 15—18

SUMMARIUM

Speculationes analyticae ab Illustr. EULERO hic traditae super formula integrali $\int \frac{x^{\alpha} - x^{\beta}}{lx}\, dx$ versantur. Cum enim eius valorem, si a termino $x=0$ usque ad terminum $x=1$ extendatur, invenisset $= l\frac{\alpha+1}{\beta+1}$, haec integratio, quippe cuius veritatem per methodos consuetas ostendere hactenus non licuerat, haud parum attentionis mereri ipsi videbatur; quamobrem considerationes, quae super hac formula Viro Illustr. sese obtulerunt, hic exponuntur variaque inde elegantissima deducuntur theoremata, quorum praecipua hic ante oculos ponemus lectorem uberioris investigationis curiosum ad ipsam dissertationem ablegantes.

Si formula $\int \frac{dx \sin. nlx}{lx}$ a termino $x=0$ usque ad terminum $x=1$ extendatur, eius valor integralis aequetur arcui circuli, cuius tangens est n; cuius theorematis veritas ex consideratione tam exponentium imaginariorum quam sequentis seriei est petenda. Cum enim sit

$$\sin. nlx = \frac{nlx}{1} - \frac{n^3(lx)^3}{1\cdot 2\cdot 3} + \frac{n^5(lx)^5}{1\cdots 5} - \frac{n^7(lx)^7}{1\cdots 7} + \text{etc.},$$

inde statim elicitur

$$\int \frac{dx \sin. nlx}{lx} = n - \frac{n^3}{3} + \frac{n^5}{5} - \frac{n^7}{7} + \frac{n^9}{9} - \text{etc.},$$

cuius seriei summa manifesto est A tang. n, ita ut posito $n=1$ fiat $\int \frac{dx \sin. lx}{lx} = \frac{\pi}{4}$ denotante π semiperipheriam circuli, cuius radius $=1$.

Si formula $\int \frac{dx \sin. plx \sin. qlx}{lx}$ a termino $x=0$ ad terminum $x=1$ extendatur, eius valor integralis deprehenditur esse $= \frac{1}{4}\, l\frac{1+(p-q)^2}{1+(p+q)^2}$, in quo igitur nullus arcus circularis occurrit, etiamsi in hoc theoremate praecedens contineri videatur. Manifestum autem est $\sin. qlx$ ad unitatem reduci non posse, nisi quantitas q variabilis accipiatur. Ex formula

autem generali quomodo huius theorematis integrale deducendum sit, investigandum indicavit Vir Illustr.; quem in finem hanc considerat formam $\int \frac{dx(x^\alpha - x^\beta)(x^\gamma - x^\delta)}{lx}$, quam in has duas resolvit $\int \frac{dx(x^{\alpha+\gamma} - x^{\beta+\gamma})}{lx} - \int \frac{dx(x^{\alpha+\delta} - x^{\beta+\delta})}{lx}$, quarum utraque cum formula generali initio memorata manifesto convenit; hinc autem labore haud operoso ad formulas in theoremate expressas pervenitur.

Si formula $\int \frac{x^m dx \sin. nlx}{lx}$ a termino $x = 0$ ad terminum $x = 1$ extendatur, ea semper huic valori $A \operatorname{tang.} \frac{n}{m+1}$ aequetur. Hic observandum est hoc theorema ad primum reduci posito $m = 1$; tum vero quoties $\frac{n}{m+1}$ eundem induit valorem, toties etiam formae integrales inter se aequales evadent.

Formula $\int \frac{x^\alpha - x^\beta}{1 + x^n} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$ semper aequetur huic formulae

$$l \frac{\alpha}{\beta} \cdot \frac{\beta + n}{\alpha + n} \cdot \frac{\alpha + 2n}{\beta + 2n} \cdot \frac{\beta + 3n}{\alpha + 3n} \cdot \text{etc.},$$

cuius producti valor per ea, quae Vir Illustr. in Miscellaneorum Berolin. Tomo VII, p. 114,[1] circa huiusmodi productum

$$\frac{a}{b} \cdot \frac{c+b}{c+a} \cdot \frac{a+k}{b+k} \cdot \frac{c+b+k}{c+a+k} \cdot \frac{a+2k}{b+2k} \cdot \frac{c+b+2k}{c+a+2k} \cdot \text{etc.}$$

docuerat, deprehenditur

$$= \frac{\int z^{n-1} dz (1 - z^{2n})^{\frac{\gamma - 2n}{2n}}}{\int z^{n-1} dz (1 - z^{2n})^{\frac{\alpha - 2n}{2n}}}.$$

Denotante i numerum infinitum formula $\int \frac{z^{\alpha i} - z^{\beta i}}{z - 1} \cdot \frac{dz}{z} \begin{bmatrix} \text{ab } z = 0 \\ \text{ad } z = 1 \end{bmatrix}$ semper est $= l \frac{\alpha}{\beta}$.

Denotantibus litteris $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. huiusmodi producta

$$\mathfrak{A} = (\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \varepsilon) \text{ etc.},$$
$$\mathfrak{B} = (\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \varepsilon) \text{ etc.},$$
$$\mathfrak{C} = (\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \varepsilon) \text{ etc.}$$
$$\text{etc.}$$

(littera vero N sit $= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n-2)$) semper erit

$$\int \frac{dx}{x(lx)^{n-1}} \left(\frac{x^\alpha}{\mathfrak{A}} + \frac{x^\beta}{\mathfrak{B}} + \frac{x^\gamma}{\mathfrak{C}} + \frac{x^\delta}{\mathfrak{D}} + \text{etc.} \right) \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix}$$
$$= \frac{\alpha^{n-2} l\alpha}{N\mathfrak{A}} + \frac{\beta^{n-2} l\beta}{N\mathfrak{B}} + \frac{\gamma^{n-2} l\gamma}{N\mathfrak{C}} + \frac{\delta^{n-2} l\delta}{N\mathfrak{D}} + \text{etc.}$$

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1) Vide notam p. 13. A. G.

Cum nuper[1]) invenissem integrale huius formulae differentialis $\frac{(x^\alpha - x^\beta)\,dx}{lx}$, si ita capiatur, ut evanescat posito $x = 0$, tum vero statuatur $x = 1$, aequari huic valori $l\frac{\alpha+1}{\beta+1}$, haec integratio eo magis attentione digna mihi videbatur, quod eius veritas per nullas methodos hactenus usitatas ostendi posset. Quamobrem nullum plane est dubium, quin ea plurimum in recessu habeat et ad multa alia praeclara inventa in Analysi perducere queat. Haud igitur ingratum Geometris fore arbitror, si nonnullas speculationes, quae super hac formula se mihi obtulerunt, exposuero.

1. Quoniam ista integratio se ad omnes plane exponentes pro litteris α et β assumtos extendit atque adeo valores imaginarii non excluduntur, ponamus

$$\alpha = n\sqrt{-1} \quad \text{et} \quad \beta = -n\sqrt{-1}$$

eritque

$$x^\alpha - x^\beta = x^{n\sqrt{-1}} - x^{-n\sqrt{-1}};$$

quae formula cum reducatur ad hanc $e^{nlx\sqrt{-1}} - e^{-nlx\sqrt{-1}}$, notum est valorem esse $= 2\sqrt{-1}\cdot \sin. nlx$, quo valore substituto prodit

$$2\sqrt{-1}\cdot\int\frac{dx \sin. nlx}{lx} = l\frac{1+n\sqrt{-1}}{1-n\sqrt{-1}}.$$

Constat autem huius formulae $l\frac{1+n\sqrt{-1}}{1-n\sqrt{-1}}$ valorem esse $2\sqrt{-1}\cdot \text{A tang.}\, n$, quandoquidem sumto n variabili eius differentiatio dat

$$d.\ l\frac{1+n\sqrt{-1}}{1-n\sqrt{-1}} = \frac{2dn\sqrt{-1}}{1+nn},$$

cuius integrale manifesto est $2\sqrt{-1}\cdot \text{A tang.}\, n$; hinc igitur adipiscimur sequens theorema:

1) Vide L. Euleri Commentationem 464 (indicis Enestroemiani): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 421, imprimis 427. A. G.

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$$\mathfrak{C}=(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\varepsilon)\ \text{etc.}$$
etc.

(littera vero N sit $=1\cdot2\cdot3\cdot4\cdot5\cdots(n-2)$) semper erit

$$\int\frac{dx}{x(lx)^{n-1}}\left(\frac{x^\alpha}{\mathfrak{A}}+\frac{x^\beta}{\mathfrak{B}}+\frac{x^\gamma}{\mathfrak{C}}+\frac{x^\delta}{\mathfrak{D}}+\text{etc.}\right)\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right]$$
$$=\frac{\alpha^{n-2}l\alpha}{N\mathfrak{A}}+\frac{\beta^{n-2}l\beta}{N\mathfrak{B}}+\frac{\gamma^{n-2}l\gamma}{N\mathfrak{C}}+\frac{\delta^{n-2}l\delta}{N\mathfrak{D}}+\text{etc.}$$

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THEOREMA 1

Ista formula integralis

$$\int \frac{dx \sin. nlx}{lx}$$

a termino $x = 0$ usque ad terminum $x = 1$ extensa exprimit arcum circuli, cuius tangens $= n$; unde sumto $n = 1$ erit

$$\int \frac{dx \sin. nlx}{lx} = \frac{\pi}{4}$$

denotante π semiperipheriam circuli, cuius radius $= 1$.

2. Quamvis autem haec integratio ex nostra forma generali, quae aliis methodis inaccessa videtur, sit deducta, tamen eius veritas per resolutiones consuetas sequenti modo ostendi potest sicque ex hoc casu integratio generalis eo maius firmamentum accipiet. Cum enim per seriem infinitam sit

$$\sin. nlx = \frac{nlx}{1} - \frac{n^3(lx)^3}{1\cdot 2\cdot 3} + \frac{n^5(lx)^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.},$$

erit

$$\int \frac{dx \sin. nlx}{lx} = \int dx\left(n - \frac{n^3(lx)^2}{1\cdot 2\cdot 3} + \frac{n^5(lx)^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.}\right);$$

constat autem esse

$$\int dx(lx)^2 = x(lx)^2 - 2\int dx lx = x(lx)^2 - 2xlx + 2\cdot 1x,$$

quae expressio posito $x = 1$ reducitur ad $2\cdot 1$; simili modo fiet

$$\int dx(lx)^4 = x(lx)^4 - 4\int dx(lx)^3 = x(lx)^4 - 4x(lx)^3 + 4\cdot 3\int dx(lx)^2,$$

quae posito $x = 1$ ob $l1 = 0$ praebet $4\cdot 3\cdot 2\cdot 1$; eodemque modo erit $\int dx(lx)^6 = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$. His igitur singulis valoribus integralibus introductis proveniet

$$\int \frac{dx \sin. nlx}{lx} = n - \frac{2\cdot 1n^3}{1\cdot 2\cdot 3} + \frac{4\cdot 3\cdot 2\cdot 1n^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{6\cdots 1n^7}{1\cdots 7} + \text{etc.}$$

$$= n - \frac{n^3}{3} + \frac{n^5}{5} - \frac{n^7}{7} + \text{etc.},$$

cuius seriei summa manifesto est A tang. n.

3. Hic casus nobis ansam praebet etiam hanc formulam integralem investigandi $\int \frac{dx \cos. nlx}{lx}$, quae quidem non immediate in nostra forma generali continetur; et quia est

$$\cos. nlx = 1 - \frac{nn(lx)^2}{1 \cdot 2} + \frac{n^4(lx)^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{n^6(lx)^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.},$$

ex primo termino oritur $\int \frac{dx}{lx}$, cuius quidem valorem ostendi esse infinitum. Pro sequentibus autem terminis erit

$$\int dx lx = xlx - x = -1 \quad \text{et} \quad \int dx(lx)^3 = -1 \cdot 2 \cdot 3 \quad \text{et} \quad \int dx(lx)^5 = -1 \cdots 5 \quad \text{etc.},$$

quibus valoribus substitutis obtinebimus

$$\int \frac{dx \cos. nlx}{lx} = \int \frac{dx}{lx} + \frac{nn}{2} - \frac{n^4}{4} + \frac{n^6}{6} - \frac{n^8}{8} + \text{etc.},$$

quae expressio manifesto reducitur ad hanc

$$\int \frac{dx}{lx} + \frac{1}{2} l(1 + nn).$$

Quia autem primus terminus hanc summam reddit infinitam, hinc subtrahamus aliam similem

$$\int \frac{dx \cos. mlx}{lx} = \int \frac{dx}{lx} + \frac{1}{2} l(1 + mm)$$

et habebimus

$$\int \frac{dx(\cos. nlx - \cos. mlx)}{lx} = \frac{1}{2} l \frac{1 + nn}{1 + mm}$$

atque haec integratio non minus notatu digna videtur quam praecedens.

4. Cum autem in genere sit $\cos. a - \cos. b = 2 \sin. \frac{b + a}{2} \sin. \frac{b - a}{2}$, erit

$$\cos. nlx - \cos. mlx = 2 \sin. \frac{m + n}{2} lx \sin. \frac{m - n}{2} lx,$$

ita ut sit

$$\int dx \frac{\sin. \frac{m + n}{2} lx \sin. \frac{m - n}{2} lx}{lx} = \frac{1}{4} l \frac{1 + nn}{1 + mm};$$

quodsi ergo ponamus $m = p + q$ et $n = p - q$, sequens adipiscemur theorema maxime notatu dignum:

THEOREMA 2

Ista forma integralis

$$\int \frac{dx}{lx} \sin. plx \sin. qlx$$

est

$$= \frac{1}{4} l \frac{1+(p-q)^2}{1+(p+q)^2},$$

si scilicet integratio a termino $x=0$ usque ad terminum $x=1$ extenditur.

Quae integratio eo magis est notatu digna, quia in ea nullus arcus circularis occurrit, etiamsi priorem in se complecti videatur, quod autem secus se habet, quia $\sin. qlx$ ad unitatem reduci nequit, quin simul quantitas q reddatur variabilis.

5. Operae igitur pretium erit investigare, quomodo etiam integrale huius theorematis ex forma nostra generali derivari queat. Hunc in finem consideremus istam formam integralem

$$\int \frac{dx}{lx}(x^\alpha - x^\beta)(x^\gamma - x^\delta),$$

quae in has duas resolvitur

$$\int \frac{dx}{lx}(x^{\alpha+\gamma} - x^{\beta+\gamma}) - \int \frac{dx}{lx}(x^{\alpha+\delta} - x^{\beta+\delta}),$$

cuius prioris valor est $l\frac{\alpha+\gamma+1}{\beta+\gamma+1}$, posterioris vero $l\frac{\alpha+\delta+1}{\beta+\delta+1}$, ita ut habeamus

$$\int \frac{dx}{lx}(x^\alpha - x^\beta)(x^\gamma - x^\delta) = l\frac{(\alpha+\gamma+1)(\beta+\delta+1)}{(\beta+\gamma+1)(\alpha+\delta+1)}.$$

Nunc igitur statuamus

$$\alpha = p\sqrt{-1} \quad \text{et} \quad \beta = -p\sqrt{-1},$$

deinde

$$\gamma = q\sqrt{-1} \quad \text{et} \quad \delta = -q\sqrt{-1},$$

ut fiat

$$x^\alpha - x^\beta = 2\sqrt{-1}\cdot \sin. plx \quad \text{et} \quad x^\gamma - x^\delta = 2\sqrt{-1}\cdot \sin. qlx;$$

sic enim nostra forma integralis induet hanc formam

$$-4\int\frac{dx}{lx}\sin.plx\sin.qlx.$$

Pro eius autem valore reperimus

$$\alpha+\gamma+1=1+(p+q)\sqrt{-1}\quad\text{et}\quad\beta+\delta+1=1-(p+q)\sqrt{-1},$$

$$\beta+\gamma+1=1+(q-p)\sqrt{-1}\quad\text{et}\quad\alpha+\delta+1=1+(p-q)\sqrt{-1},$$

quibus valoribus substitutis valor integralis prodit

$$l\frac{1+(p+q)^2}{1+(p-q)^2}=-l\frac{1+(p-q)^2}{1+(p+q)^2},$$

unde manifesto sequitur integratio postremi theorematis

$$\int\frac{dx}{lx}\sin.plx\sin.qlx=\frac{1}{4}l\frac{1+(p-q)^2}{1+(p+q)^2}.$$

6. Hinc occasionem arripimus etiam hanc formam generalem evolvendi

$$\int\frac{dx}{lx}(x^\alpha-x^\beta)(x^\gamma+x^\delta),$$

cuius valor erit

$$=l\frac{\alpha+\gamma+1}{\beta+\gamma+1}+l\frac{\alpha+\delta+1}{\beta+\delta+1}=l\frac{(\alpha+\gamma+1)(\alpha+\delta+1)}{(\beta+\gamma+1)(\beta+\delta+1)}.$$

Nunc iterum faciamus

$$\alpha=p\sqrt{-1}\quad\text{et}\quad\beta=-p\sqrt{-1},$$

tum vero

$$\gamma=q\sqrt{-1}\quad\text{et}\quad\delta=-q\sqrt{-1}$$

fietque

$$x^\alpha-x^\beta=2\sqrt{-1}\cdot\sin.plx\quad\text{et}\quad x^\gamma+x^\delta=2\cos.qlx,$$

ita ut ipsa formula integralis oriatur

$$4\sqrt{-1}\cdot\int\frac{dx}{lx}\sin.plx\cos.qlx.$$

Pro valore autem integrali habebimus

$$\alpha+\gamma+1=1+(p+q)\sqrt{-1},\quad \beta+\gamma+1=1+(q-p)\sqrt{-1},$$
$$\alpha+\delta+1=1+(p-q)\sqrt{-1},\quad \beta+\delta+1=1-(p+q)\sqrt{-1},$$

unde valor integralis prodit

$$l\frac{1+(p+q)\sqrt{-1}}{1-(p+q)\sqrt{-1}}\cdot\frac{1+(p-q)\sqrt{-1}}{1-(p-q)\sqrt{-1}}=l\frac{1+(p+q)\sqrt{-1}}{1-(p+q)\sqrt{-1}}+l\frac{1+(p-q)\sqrt{-1}}{1-(p-q)\sqrt{-1}}.$$

Est vero

$$l\frac{1+(p+q)\sqrt{-1}}{1-(p+q)\sqrt{-1}}=2\sqrt{-1}\cdot \text{A tang.}\,(p+q)$$

eodemque modo

$$l\frac{1+(p-q)\sqrt{-1}}{1-(p-q)\sqrt{-1}}=2\sqrt{-1}\cdot \text{A tang.}\,(p-q),$$

quibus valoribus substitutis resultat ista integratio

$$\int\frac{dx}{lx}\sin.plx\cos.qlx=\frac{1}{2}\text{A tang.}\,(p+q)+\frac{1}{2}\text{A tang.}\,(p-q).$$

Cum igitur sit in genere $\text{A tang.}\,a+\text{A tang.}\,b=\text{A tang.}\,\frac{a+b}{1-ab}$, erit summa arcuum modo inventa $=\text{A tang.}\,\frac{2p}{1-pp+qq}$ et valor integralis

$$\frac{1}{2}\text{A tang.}\,\frac{2p}{1-pp+qq};$$

hinc sequens

THEOREMA 3

Ista formula integralis

$$\int\frac{dx}{lx}\sin.plx\cos.qlx$$

a termino $x=0$ usque ad $x=1$ extensa aequalis est huic valori

$$\frac{1}{2}\text{A tang.}\,\frac{2p}{1-pp+qq}.$$

7. Quodsi ergo sumamus $q = p$, ob $\sin. plx \cos. qlx = \frac{1}{2} \sin. 2plx$ prodibit ista integratio

$$\frac{1}{2}\int \frac{dx}{lx} \sin. 2plx = \frac{1}{2} \text{A} \tang. 2p,$$

id quod prorsus convenit cum theoremate primo; at vero etiam in genere ad theorema primum reduci potest. Cum enim sit

$$\sin. a \cos. b = \frac{1}{2} \sin. (a+b) + \frac{1}{2} \sin. (a-b),$$

formula nostra in has partes dividitur

$$\frac{1}{2}\int \frac{dx}{lx} \sin. (p+q) lx + \frac{1}{2}\int \frac{dx}{lx} \sin. (p-q) lx.$$

Prioris igitur partis valor erit ex theoremate

$$= \frac{1}{2} \text{A} \tang. (p+q),$$

posterioris vero partis

$$= \frac{1}{2} \text{A} \tang. (p-q),$$

quae forma utique reducitur ad eam, quam hic dedimus.

8. Nunc autem integrationem nostram generalem

$$\int \frac{dx}{lx} (x^{\alpha} - x^{\beta}) = l \frac{\alpha+1}{\beta+1}$$

aliquanto generalius ad angulos reducamus ponendo

$$\alpha = m + n\sqrt{-1}, \quad \beta = m - n\sqrt{-1},$$

ut fiat

$$x^{\alpha} - x^{\beta} = x^{m}(x^{n\sqrt{-1}} - x^{-n\sqrt{-1}}) = 2\sqrt{-1} \cdot x^{m} \sin. nlx;$$

tum vero erit

$$\frac{\alpha+1}{\beta+1} = \frac{1+m+n\sqrt{-1}}{1+m-n\sqrt{-1}},$$

quae fractio posito $n = k(m+1)$ reducitur ad hanc $\frac{1+k\sqrt{-1}}{1-k\sqrt{-1}}$. Est vero

$$l\frac{1+k\sqrt{-1}}{1-k\sqrt{-1}} = 2\sqrt{-1}\cdot \text{A tang.}\, k = 2\sqrt{-1}\cdot \text{A tang.}\,\frac{n}{m+1}$$

sicque impetramus sequens theorema:

THEOREMA 4

Ista formula integralis

$$\int \frac{dx}{lx} x^m \sin.\, nlx$$

a termino $x = 0$ *usque ad terminum* $x = 1$ *extensa semper aequalis erit huic valori*

$$\text{A tang.}\,\frac{n}{m+1}.$$

Quod theorema sumto $m = 0$ ad primum reducitur; ubi imprimis notatu dignum occurrit, quod, quoties $\frac{n}{m+1}$ eundem habet valorem, toties etiam formae integrales aequales inter se evadunt.

9. Verum etiam hoc theorema in genere ad primum reduci potest. Si enim ponatur $x^{m+1} = y$, erit

$$x^m dx = \frac{dy}{m+1} \quad \text{et} \quad lx = \frac{ly}{m+1};$$

his valoribus substitutis fiet

$$\int \frac{dy}{ly} \sin.\, \frac{n}{m+1} ly;$$

quae cum similis sit primo theoremati, eius valor manifesto est $= \text{A tang.}\,\frac{n}{m+1}$; quoniam autem hic posuimus $x^{m+1} = y$, ambo termini integrationis hic etiam erunt $y = 0$ et $y = 1$.

10. Per hoc ergo theorema, cum sit

$$\text{A tang.}\,\frac{1}{2} + \text{A tang.}\,\frac{1}{3} = \text{A tang.}\,1 = \frac{\pi}{4},$$

pro priore erit $n = 1$ et $m = 1$, pro posteriore vero $n = 1$ et $m = 2$; hinc igitur habebitur ista integratio

$$\int \frac{dx}{lx}(x + xx) \sin. lx = \frac{\pi}{4}.$$

Deinde cum per seriem infinitam sit

$$\frac{\pi}{4} = \text{A tang.} \frac{1}{2} + \text{A tang.} \frac{1}{8} + \text{A tang.} \frac{1}{18} + \text{A tang.} \frac{1}{32} + \text{A tang.} \frac{1}{50} + \text{etc.},$$

cuius seriei terminus generalis est $\text{A tang.} \frac{1}{2nn}$, habebimus hanc integrationem satis memorabilem

$$\int \frac{dx}{xlx}(x^2 + x^8 + x^{18} + x^{32} + \text{etc.}) \sin. lx = \frac{\pi}{4},$$

quod eo magis est notatu dignum, quod series infinita $x^2 + x^8 + x^{18} + x^{32} + \text{etc.}$ nullo modo ad summam finitam reduci potest.

11. Sed revertamur ad nostram integrationem principalem, qua est

$$\int \frac{dx}{lx}(x^\alpha - x^\beta) = l\frac{\alpha + 1}{\beta + 1},$$

cuius veritatem etiam hoc modo ostendere licet. Cum sit $x^\alpha = e^{\alpha lx}$ denotante e numerum, cuius logarithmus hyperbolicus $= 1$, erit per seriem infinitam

$$x^\alpha = 1 + \frac{\alpha lx}{1} + \frac{\alpha\alpha(lx)^2}{1 \cdot 2} + \frac{\alpha^3(lx)^3}{1 \cdot 2 \cdot 3} + \frac{\alpha^4(lx)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

hincque colligitur fore

$$x^\alpha - x^\beta = (\alpha - \beta)\frac{lx}{1} + (\alpha\alpha - \beta\beta)\frac{(lx)^2}{1 \cdot 2} + (\alpha^3 - \beta^3)\frac{(lx)^3}{1 \cdot 2 \cdot 3} + \text{etc.},$$

quae series per $\frac{dx}{lx}$ multiplicata et integrata ob

$$\int dx(lx)^n = \pm 1 \cdot 2 \cdot 3 \cdots n$$

(ubi signum superius valet, si n est numerus par, inferius, si impar) praebet

posito $x=1$ sequentem seriem

$$\alpha-\beta-\frac{\alpha^2-\beta^2}{2}+\frac{\alpha^3-\beta^3}{3}-\frac{\alpha^4-\beta^4}{4}+\text{etc.},$$

quae series manifesto praebet

$$l(1+\alpha)-l(1+\beta)=l\frac{\alpha+1}{\beta+1}.$$

12. Quo valor huius formulae succinctius exprimatur, loco α et β scribamus $\alpha-1$ et $\beta-1$, ut habeamus

$$\int\frac{dx}{lx}(x^{\alpha-1}-x^{\beta-1})=\int\frac{dx}{xlx}(x^\alpha-x^\beta)=l\frac{\alpha}{\beta}.$$

Quodsi ergo sumamus $\alpha=e^m$ et $\beta=e^n$, nanciscemur sequentem integrationem satis concinnam

$$\int\frac{dx}{xlx}(x^{e^m}-x^{e^n})=m-n.$$

13. Investigemus nunc integrale huius formulae differentialis

$$\frac{dx}{xlx}\cdot\frac{x^\alpha-x^\beta}{1+x^n},$$

et cum sit

$$\frac{1}{1+x^n}=1-x^n+x^{2n}-x^{3n}+x^{4n}-\text{etc.},$$

colligitur hinc integrale quaesitum

$$l\frac{\alpha}{\beta}-l\frac{\alpha+n}{\beta+n}+l\frac{\alpha+2n}{\beta+2n}-l\frac{\alpha+3n}{\beta+3n}+\text{etc.},$$

unde nanciscimur sequens theorema:

THEOREMA 5

Ista formula integralis

$$\int\frac{dx}{xlx}\cdot\frac{x^\alpha-x^\beta}{1+x^n}$$

a termino $x=0$ usque ad terminum $x=1$ extensa semper aequatur huic formulae logarithmicae

$$l\frac{\alpha}{\beta}\cdot\frac{\beta+n}{\alpha+n}\cdot\frac{\alpha+2n}{\beta+2n}\cdot\frac{\beta+3n}{\alpha+3n}\cdot\frac{\alpha+4n}{\beta+4n}\cdot\text{etc.}$$

14. Cum igitur alibi[1]) demonstraverim huius producti in infinitum continuati

$$\frac{a}{b}\cdot\frac{c+b}{c+a}\cdot\frac{a+k}{b+k}\cdot\frac{c+b+k}{c+a+k}\cdot\frac{a+2k}{b+2k}\cdot\frac{c+b+2k}{c+a+2k}\cdot\text{ etc.}$$

valorem aequari huic expressioni

$$\frac{\int z^{c-1}dz(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}dz(1-z^k)^{\frac{a-k}{k}}},$$

applicatione ad nostrum casum facta erit

$$a=\alpha,\quad b=\beta,\quad c=n,\quad k=2n$$

hincque valor nostri producti infiniti

$$=\frac{\int z^{n-1}dz(1-z^{2n})^{\frac{\beta-2n}{2n}}}{\int z^{n-1}dz(1-z^{2n})^{\frac{\alpha-2n}{2n}}},$$

quae ambae formulae integrales a termino $z=0$ usque ad terminum $z=1$ sunt extendendae; atque hinc colligimus sequens theorema:

THEOREMA 6

Ista formula integralis

$$\int\frac{dx}{xlx}\cdot\frac{x^\alpha-x^\beta}{1+x^n}$$

a termino $x=0$ usque ad terminum $x=1$ extensa aequalis est huic valori

$$l\frac{P}{Q}$$

existente

$$P=\int z^{n-1}dz(1-z^{2n})^{\frac{\beta-2n}{2n}}\quad et\quad Q=\int z^{n-1}dz(1-z^{2n})^{\frac{\alpha-2n}{2n}},$$

dum scilicet etiam hae formulae integrales posteriores a termino $z=0$ usque ad terminum $z=1$ extenduntur.

1) Vide L. Euleri Commentationes 59 et 122 (indicis Enestroemiani): *Theoremata circa reductionem formularum integralium ad quadraturam circuli,* Miscellanea Berolin. 7, 1743, p. 91, et *De productis ex infinitis factoribus ortis,* Comment. acad. sc. Petrop. **11** (1739), 1750, p. 3; *Leonhardi Euleri Opera omnia,* series I, vol. 17 et 14. A. G.

15. Sumamus igitur $n=1$, ut formula nostra integralis fiat

$$\int \frac{dx}{xlx}\cdot\frac{x^\alpha-x^\beta}{1+x},$$

ac tum erit

$$P=\int dz(1-zz)^{\frac{\beta-2}{2}} \quad \text{et} \quad Q=\int dz(1-zz)^{\frac{\alpha-2}{2}},$$

unde pro α et β sequentes casus evolvamus. Sit primo $\alpha=2$ et $\beta=1$; erit $P=\text{A}\sin.z=\frac{\pi}{2}$ et $Q=z=1$ ideoque $\frac{P}{Q}=\frac{\pi}{2}$, unde colligimus fore

$$\int \frac{dx}{lx}\cdot\frac{x-1}{x+1}=l\frac{\pi}{2}.$$

16. Sumamus nunc $\alpha=3$ et $\beta=1$, ut fiat $\frac{x^\alpha-x^\beta}{1+x}=x(x-1)$, hincque formula nostra integralis erit $\int\frac{dx}{lx}(x-1)$, cuius valorem novimus esse $=l2$; at vero ex formula nostra generali erit

$$P=\frac{\pi}{2} \quad \text{et} \quad Q=\int dz\sqrt{(1-zz)}=\int\frac{dz}{\sqrt{(1-zz)}}-\int\frac{zzdz}{\sqrt{(1-zz)}}.$$

At vero per reductiones notas est

$$\int\frac{zzdz}{\sqrt{(1-zz)}}=\frac{1}{2}\int\frac{dz}{\sqrt{(1-zz)}}$$

sicque erit

$$Q=\frac{1}{2}\int\frac{dz}{\sqrt{(1-zz)}}=\frac{1}{2}\cdot\frac{\pi}{2},$$

unde fit $\frac{P}{Q}=2$, qui valor perfecte congruit cum ante assignato.

17. Quoniam in quantitate P non occurrit exponens α, in altero vero Q tantum α occurrit, superius theorema ita in duas partes distribuere licebit, ut sit

$$\int\frac{dx}{lx}\cdot\frac{x^{\alpha-1}}{1+x^n}=C-l\int z^{n-1}dz(1-z^{2n})^{\frac{\alpha-2n}{2n}}$$

et

$$\int\frac{dx}{lx}\cdot\frac{x^{\beta-1}}{1+x^n}=C-l\int z^{n-1}dz(1-z^{2n})^{\frac{\beta-2n}{2n}},$$

ubi C denotant certam constantem, quae autem in differentia duarum huiusmodi formularum integralium e calculo egreditur.

18. Possumus etiam nostram formulam integralem principalem

$$\int \frac{dx}{xlx}(x^\alpha - x^\beta) = l\frac{\alpha}{\beta}$$

ita transformare, ut in ea exponentes infiniti occurrant, quae ob hoc ipsum attentione non indigna videtur. Denotet igitur i numerum infinite magnum, et quia lx ita exprimere licet, ut sit $lx = i(x^{\frac{1}{i}} - 1)$, formula nostra hanc induet formam

$$\int \frac{dx}{ix(x^{\frac{1}{i}} - 1)}(x^\alpha - x^\beta) = l\frac{\alpha}{\beta}.$$

Nunc igitur ad exponentem fractum tollendum statuamus $x^{\frac{1}{i}} = z$, ut sit $x = z^i$ hincque $\frac{dx}{x} = \frac{idz}{z}$; tum vero $x^\alpha = z^{\alpha i}$ et $x^\beta = z^{\beta i}$, et quia adhuc iidem termini integrationis habentur $z = 0$ et $z = 1$, hinc sequens theorema resultat:

THEOREMA 7

Denotante i numerum infinite magnum ista formula integralis

$$\int \frac{dz(z^{\alpha i} - z^{\beta i})}{z(z-1)}$$

a termino $z = 0$ usque ad terminum $z = 1$ extensa semper aequalis est huic valori

$$l\frac{\alpha}{\beta}.$$

19. Cum sit

$$\frac{z^{\alpha i}}{z-1} = z^{\alpha i - 1} + z^{\alpha i - 2} + z^{\alpha i - 3} + z^{\alpha i - 4} + \text{etc.},$$

erit

$$\int \frac{z^{\alpha i}dz}{z(z-1)} = \frac{1}{\alpha i - 1} + \frac{1}{\alpha i - 2} + \frac{1}{\alpha i - 3} + \frac{1}{\alpha i - 4} + \text{etc.} + C$$

eodemque modo erit

$$\int \frac{z^{\beta i}dz}{z(z-1)} = \frac{1}{\beta i - 1} + \frac{1}{\beta i - 2} + \frac{1}{\beta i - 3} + \frac{1}{\beta i - 4} + \text{etc.} + C,$$

unde patet differentiam harum duarum serierum esse $l\frac{\alpha}{\beta}$; ita si fuerit $\alpha = 2$ et $\beta = 1$, prodibit ista integratio

$$\int \frac{dz(z^{2i}-z^{i})}{z(z-1)} = \frac{1}{2i-1} + \frac{1}{2i-2} + \frac{1}{2i-3} + \frac{1}{2i-4} + \cdots + \frac{1}{i},$$

quoniam sequentes termini per seriem posteriorem tolluntur. Constat autem huius seriei summam esse $l2$.

20. Plurima adhuc alia consectaria ex ista integratione memorabili deduci possent, quibus autem hic non immorabor, sed potius ipsam analysin, quae ad hanc integrationem perduxit, accuratius perpendam. Consideravi scilicet potestatem x^u, cuius exponens u pro lubitu sive ut constans sive ut variabilis spectari queat, et cum sit

$$\int \frac{x^u dx}{x} = \frac{x^u}{u},$$

ideoque si post integrationem sumatur $x = 1$, erit $\int \frac{dx}{x} x^u = \frac{1}{u}$; quae formula ergo fundamentum constituat, unde sequentia deducemus.

21. Hanc iam formulam per du multiplicatam integremus spectata x ut constante, et quia summa $\int x^u du = \frac{x^u}{lx}$, tum vero constat hanc integrationem ab altera, ubi x erat variabilis, non turbari, habebimus nunc istam integrationem

$$\int \frac{dx}{xlx} x^u = lu + A,$$

ubi A denotat constantem per integrationem ingressam, quae igitur e medio tolletur, si duas huiusmodi formas a se invicem subtrahamus; unde si primo sumamus $u = \alpha$, tum vero $u = \beta$ et posterius integrale a priori subtrahamus, prodibit nostra forma principalis initio commemorata

$$\int \frac{dx}{xlx}(x^\alpha - x^\beta) = l\frac{\alpha}{\beta}.$$

22. Simili autem modo a formula integrali $\int \frac{dx}{xlx} x^u = lu + A$ ulterius progrediamur, qua per du multiplicata et ex sola variabilitate ipsius u integrata

ob $\int x^u du = \frac{x^u}{lx}$ ut ante perveniemus ad istam integrationem

$$\int \frac{dx}{x(lx)^2} x^u = \int du lu + Au + B = ulu - u + Au + B,$$

ubi ergo ternas formulas particulares inter se combinari oportet, ut ambae constantes A et B ex calculo deturbentur; quia autem loco A scribere licet $A+1$, erit

$$\int \frac{dx}{x(lx)^2} x^u = ulu + Au + B.$$

23. Quodsi iam denuo per du multiplicemus et integremus, mutatis litteris constantibus, quo formula concinnior reddatur, reperiemus

$$\int \frac{dx}{x(lx)^3} x^u = \frac{1}{2} uulu + Auu + Bu + C$$

eodemque modo ulterius

$$\int \frac{dx}{x(lx)^4} x^u = \frac{1}{6} u^3 lu + Au^3 + Buu + Cu + D,$$

$$\int \frac{dx}{x(lx)^5} x^u = \frac{1}{24} u^4 lu + Au^4 + Bu^3 + Cuu + Du + E$$

etc.

Unde intelligitur continuo plures casus particulares invicem coniungi debere, ut omnes quantitates constantes A, B, C, D etc. ex calculo expellantur.

24. Hoc igitur modo evolvamus formulam § 22 inventam et exponenti u tribuamus hos tres valores α, β et γ, ut obtineamus istas tres formulas

$$\text{I.} \quad \int \frac{dx}{x(lx)^2} x^\alpha = \alpha l\alpha + A\alpha + B,$$

$$\text{II.} \quad \int \frac{dx}{x(lx)^2} x^\beta = \beta l\beta + A\beta + B,$$

$$\text{III.} \quad \int \frac{dx}{x(lx)^2} x^\gamma = \gamma l\gamma + A\gamma + B,$$

unde eliminando B duas hasce nanciscimur aequationes

$$\mathrm{I}-\mathrm{II}=\alpha l\alpha-\beta l\beta+A(\alpha-\beta) \quad \text{et} \quad \mathrm{II}-\mathrm{III}=\beta l\beta-\gamma l\gamma+A(\beta-\gamma),$$

$$(\mathrm{I}-\mathrm{II})(\beta-\gamma)-(\mathrm{II}-\mathrm{III})(\alpha-\beta)$$
$$=(\beta-\gamma)\alpha l\alpha-(\beta-\gamma)\beta l\beta-(\alpha-\beta)\beta l\beta+(\alpha-\beta)\gamma l\gamma,$$

quae reducitur ad hanc

$$\mathrm{I}(\beta-\gamma)+\mathrm{II}(\gamma-\alpha)+\mathrm{III}(\alpha-\beta)=(\beta-\gamma)\alpha l\alpha+(\gamma-\alpha)\beta l\beta+(\alpha-\beta)\gamma l\gamma.$$

25. Hinc igitur pro formulis ad istud genus referendis constituere poterimus sequens theorema fundamentale:

THEOREMA 8

Ista formula integralis

$$\int\frac{dx}{x(lx)^2}((\beta-\gamma)x^\alpha+(\gamma-\alpha)x^\beta+(\alpha-\beta)x^\gamma)$$

a termino $x=0$ usque ad terminum $x=1$ extensa semper aequalis est huic valori

$$(\beta-\gamma)\alpha l\alpha+(\gamma-\alpha)\beta l\beta+(\alpha-\beta)\gamma l\gamma.$$

26. Circa hanc formam imprimis notasse iuvabit formulam

$$(\beta-\gamma)x^\alpha+(\gamma-\alpha)x^\beta+(\alpha-\beta)x^\gamma$$

non solum per $x-1$ esse divisibilem, sed etiam per $(x-1)^2$; prius inde patet, quod posito $x=1$ fit $\beta-\gamma+\gamma-\alpha+\alpha-\beta=0$, posterius vero, quod eius etiam differentiale posito $x=1$ fit $\alpha(\beta-\gamma)+\beta(\gamma-\alpha)+\gamma(\alpha-\beta)=0$, id quod natura rei postulat, quia in denominatore $(lx)^2$ posito $x=1$ continetur quadratum quantitatis evanescentis.

27. Quo vis huius integrationis generalis clarius perspiciatur, casum evolvisse operae pretium erit, quo ponitur $\alpha=n+2$, $\beta=n+1$ et $\gamma=n$, quandoquidem obtinebitur ista integratio

$$\int\frac{x^{n-1}dx(x-1)^2}{(lx)^2}=(n+2)l(n+2)-2(n+1)l(n+1)+nln=l\frac{(n+2)^{n+2}n^n}{(n+1)^{2(n+1)}}.$$

28. Tractemus eodem modo formulam integralem gradus tertii, in qua occurrit $(lx)^3$, tribuendo exponenti u quatuor valores $\alpha, \beta, \gamma, \delta$, unde oriuntur sequentes aequationes

$$\text{I.} \quad \int \frac{dx}{x(lx)^3} x^\alpha = \frac{1}{2}\alpha\alpha l\alpha + A\alpha\alpha + B\alpha + C,$$

$$\text{II.} \quad \int \frac{dx}{x(lx)^3} x^\beta = \frac{1}{2}\beta\beta l\beta + A\beta\beta + B\beta + C,$$

$$\text{III.} \quad \int \frac{dx}{x(lx)^3} x^\gamma = \frac{1}{2}\gamma\gamma l\gamma + A\gamma\gamma + B\gamma + C,$$

$$\text{IV.} \quad \int \frac{dx}{x(lx)^3} x^\delta = \frac{1}{2}\delta\delta l\delta + A\delta\delta + B\delta + C.$$

Atque hinc erit primo

$$\text{I} - \text{II} = \frac{1}{2}\alpha\alpha l\alpha - \frac{1}{2}\beta\beta l\beta + A(\alpha\alpha - \beta\beta) + B(\alpha - \beta),$$

unde fit

$$\frac{\text{I} - \text{II}}{\alpha - \beta} = \frac{\alpha\alpha l\alpha - \beta\beta l\beta}{2(\alpha - \beta)} + A(\alpha + \beta) + B.$$

Eodemque modo erit

$$\frac{\text{II} - \text{III}}{\beta - \gamma} = \frac{\beta\beta l\beta - \gamma\gamma l\gamma}{2(\beta - \gamma)} + A(\beta + \gamma) + B,$$

quarum formularum differentia dat

$$\frac{\text{I} - \text{II}}{\alpha - \beta} - \frac{\text{II} - \text{III}}{\beta - \gamma} = \frac{\alpha\alpha l\alpha - \beta\beta l\beta}{2(\alpha - \beta)} - \frac{\beta\beta l\beta - \gamma\gamma l\gamma}{2(\beta - \gamma)} + A(\alpha - \gamma),$$

qua per $\alpha - \gamma$ divisa prodit

$$\frac{\text{I} - \text{II}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{\text{II} - \text{III}}{(\beta - \gamma)(\alpha - \gamma)} = \frac{\alpha\alpha l\alpha - \beta\beta l\beta}{2(\alpha - \beta)(\alpha - \gamma)} - \frac{\beta\beta l\beta - \gamma\gamma l\gamma}{2(\beta - \gamma)(\alpha - \gamma)} + A;$$

eodemque modo erit

$$\frac{\text{II} - \text{III}}{(\beta - \gamma)(\beta - \delta)} - \frac{\text{III} - \text{IV}}{(\gamma - \delta)(\beta - \delta)} = \frac{\beta\beta l\beta - \gamma\gamma l\gamma}{2(\beta - \gamma)(\beta - \delta)} - \frac{\gamma\gamma l\gamma - \delta\delta l\delta}{2(\gamma - \delta)(\beta - \delta)} + A,$$

quae postrema a superiori sublata relinquit

$$\frac{\mathrm{I}-\mathrm{II}}{(\alpha-\beta)(\alpha-\gamma)}-\frac{\mathrm{II}-\mathrm{III}}{(\beta-\gamma)(\alpha-\gamma)}-\frac{\mathrm{II}-\mathrm{III}}{(\beta-\gamma)(\beta-\delta)}+\frac{\mathrm{III}-\mathrm{IV}}{(\gamma-\delta)(\beta-\delta)}$$
$$=\frac{\alpha\alpha l\alpha-\beta\beta l\beta}{2(\alpha-\beta)(\alpha-\gamma)}-\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)(\alpha-\gamma)}-\frac{\beta\beta l\beta-\gamma\gamma l\gamma}{2(\beta-\gamma)(\beta-\delta)}+\frac{\gamma\gamma l\gamma-\delta\delta l\delta}{2(\gamma-\delta)(\beta-\delta)};$$

sicque iam omnes tres constantes A, B, C sunt elisae.

29. Quodsi iam singula huius aequationis membra evolvantur et tam secundum numeros I, II, III, IV quam secundum formulas $\alpha\alpha l\alpha$, $\beta\beta l\beta$, $\gamma\gamma l\gamma$, $\delta\delta l\delta$ in ordinem disponantur, obtinebitur sequens aequatio

$$\frac{\mathrm{I}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\mathrm{II}(\alpha-\delta)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\mathrm{III}(\alpha-\delta)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\mathrm{IV}(\alpha-\delta)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$
$$=\frac{\alpha\alpha l\alpha}{2(\alpha-\beta)(\alpha-\gamma)}+\frac{(\alpha-\delta)\beta\beta l\beta}{2(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{(\alpha-\delta)\gamma\gamma l\gamma}{2(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{(\alpha-\delta)\delta\delta l\delta}{2(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$

quae aequatio per $\alpha-\delta$ divisa ad pulcherrimam uniformitatem reducitur; quo facto sequens nanciscimur theorema ad hunc casum accommodatum:

THEOREMA 9

Ista formula integralis

$$\int\frac{dx}{x(lx)^3}\left\{\begin{array}{l}\dfrac{x^\alpha}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\dfrac{x^\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\\+\dfrac{x^\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\dfrac{x^\delta}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\end{array}\right\}$$

a termino $x=0$ usque ad terminum $x=1$ extensa aequatur sequenti formulae

$$\frac{\alpha\alpha l\alpha}{2(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta\beta l\beta}{2(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}$$
$$+\frac{\gamma\gamma l\gamma}{2(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta\delta l\delta}{2(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

Ex qua forma perspicitur, quomodo ad casus magis compositos facile progredi liceat.

30. Ad hunc modum etiam praecedentes casus repraesentare operae pretium erit. Ita pro divisore lx habebimus sequentem formam integralem

$$\int \frac{dx}{x\,lx}\left(\frac{x^{\alpha}}{\alpha-\beta}+\frac{x^{\beta}}{\beta-\alpha}\right)=\frac{l\alpha}{\alpha-\beta}+\frac{l\beta}{\beta-\alpha}.$$

Deinde theorema § 24 allatum ita referetur

$$\int \frac{dx}{x(lx)^2}\left(\frac{x^{\alpha}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{x^{\beta}}{(\beta-\alpha)(\beta-\gamma)}+\frac{x^{\gamma}}{(\gamma-\alpha)(\gamma-\beta)}\right)$$
$$=\frac{\alpha l\alpha}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta l\beta}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma l\gamma}{(\gamma-\alpha)(\gamma-\beta)}$$

atque istam formam sequitur illa, quam in theoremate ultimo retulimus.

31. Nunc igitur hoc negotium in genere expedire poterimus pro quacunque potestate ipsius lx, quae in denominatore formulae integralis occurrit, cuius exponens sit $=n-1$, ut numerus membrorum fiat $=n$; tum igitur accipiantur pro lubitu numeri α, β, γ, δ etc., quorum numerus sit $=n$, et quaerantur hinc sequentes valores

$$\mathfrak{A}=(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)\text{ etc.},$$
$$\mathfrak{B}=(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\varepsilon)\text{ etc.},$$
$$\mathfrak{C}=(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\varepsilon)\text{ etc.},$$
$$\mathfrak{D}=(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\varepsilon)\text{ etc.}$$
$$\text{etc.},$$

tum vero ponatur etiam brevitatis gratia hoc productum

$$1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(n-2)=N$$

atque obtinebitur sequens forma integralis generalissima

$$\int \frac{dx}{x(lx)^{n-1}}\left(\frac{x^{\alpha}}{\mathfrak{A}}+\frac{x^{\beta}}{\mathfrak{B}}+\frac{x^{\gamma}}{\mathfrak{C}}+\frac{x^{\delta}}{\mathfrak{D}}+\text{etc.}\right)=\frac{\alpha^{n-2}l\alpha}{N\mathfrak{A}}+\frac{\beta^{n-2}l\beta}{N\mathfrak{B}}+\frac{\gamma^{n-2}l\gamma}{N\mathfrak{C}}+\frac{\delta^{n-2}l\delta}{N\mathfrak{D}}+\text{etc.},$$

ubi notandum casu $n=2$ fore $N=1$.

32. Ad haec uberius illustranda meminisse iuvabit me iam pridem[1]) insigne theorema arithmeticum demonstrasse circa huiusmodi fractiones $\frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} +$ etc., quorum numerus sit ut ante $= n$, ubi ostendi omnes sequentes formulas nihilo aequari:

$$\text{I.} \quad \frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} + \frac{1}{\mathfrak{D}} + \text{etc.} = 0,$$

$$\text{II.} \quad \frac{\alpha}{\mathfrak{A}} + \frac{\beta}{\mathfrak{B}} + \frac{\gamma}{\mathfrak{C}} + \frac{\delta}{\mathfrak{D}} + \text{etc.} = 0,$$

$$\text{III.} \quad \frac{\alpha\alpha}{\mathfrak{A}} + \frac{\beta\beta}{\mathfrak{B}} + \frac{\gamma\gamma}{\mathfrak{C}} + \frac{\delta\delta}{\mathfrak{D}} + \text{etc.} = 0,$$

$$\text{IV.} \quad \frac{\alpha^3}{\mathfrak{A}} + \frac{\beta^3}{\mathfrak{B}} + \frac{\gamma^3}{\mathfrak{C}} + \frac{\delta^3}{\mathfrak{D}} + \text{etc.} = 0$$

etc.,

donec perveniatur ad potestatem exponentis $n - 2$; at vero sumto exponente $= n - 1$ semper fore demonstravimus

$$\frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \frac{\delta^{n-1}}{\mathfrak{D}} + \text{etc.} = 1.$$

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1) Vide epistolas d. 25. Sept. et 9. Nov. 1762 ab EULERO ad CHR. GOLDBACH scriptas, *Correspondance math. et phys. publiée par P. H. Fuss*, St.-Pétersbourg 1843, t. I, p. 659 et 663; *LEONHARDI EULERI Opera omnia*, series III. Vide porro L. EULERI *Institutionum calculi integralis* vol. II, § 1169, Petropoli 1769; *LEONHARDI EULERI Opera omnia*, series I, vol. 12, p. 341. Vide praeterea EULERI Commentationem 794 (indicis ENESTROEMIANI): *Theorema arithmeticum eiusque demonstratio*, *Comment. arithm.* 2, 1849, p. 588; *LEONHARDI EULERI Opera omnia*, series I, vol. 6.
A. G.

DE INTEGRATIONE FORMULAE

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}}$$

AB $x=0$ AD $x=1$ EXTENSA

Commentatio 499 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1777: II, 1780, p. 3—28

1. Methodus maxime naturalis huiusmodi formulas $\int p\,dx\,lx$ tractandi in hoc consistit, ut eae ad alias huiusmodi formas $\int q\,dx$ reducantur, in quibus littera q sit functio algebraica ipsius x, quandoquidem regulae integrandi potissimum ad tales formulas sunt accommodatae. Huiusmodi autem reductio nulla prorsus laborat difficultate, quando functio p ita est comparata, ut integrale $\int p\,dx$ algebraice exhiberi queat. Si enim fuerit $\int p\,dx = P$, ita ut formula proposita sit $\int dP\,lx$, ea sponte reducitur ad hanc expressionem

$$Plx - \int \frac{P\,dx}{x}$$

sicque iam totum negotium ad integrationem huius formulae $\int \frac{P\,dx}{x}$ est perductum. Quando vero formula $\int p\,dx$ integrationem algebraicam non admittit, quemadmodum evenit in nostra formula proposita $\int \frac{dx\,lx}{\sqrt{1-xx}}$, talis reductio successu penitus caret. Cum enim sit $\int \frac{dx}{\sqrt{1-xx}} = \mathrm{A}\sin.x$, ista reductio daret

$$\int \frac{dx\,lx}{\sqrt{1-xx}} = \mathrm{A}\sin.x\,lx - \int \frac{dx}{x}\,\mathrm{A}\sin.x$$

sicque post signum integrationis nova quantitas transcendens $\mathrm{A}\sin.x$ occur-

reret, cuius integratio aeque est abscondita ac ipsius propositae. Quare cum nuper singulari methodo invenissem esse

$$\int \frac{dxlx}{\sqrt{(1-xx)}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\pi}{2} l2,$$

expressio integralis eo maiori attentione digna est censenda, quod eius investigatio neutiquam est obvia; unde operae pretium esse duxi eius veritatem etiam ex aliis fontibus ostendisse, antequam ipsam methodum, quae me eo perduxit, exponerem.

PRIMA DEMONSTRATIO INTEGRATIONIS PROPOSITAE

2. Quoniam hic potissimum ad series infinitas est recurrendum, formula autem lx talem resolutionem simplicem respuit, adhibeamus substitutionem $\sqrt{(1-xx)}=y$, unde fit $x=\sqrt{1-yy}$ hincque porro

$$lx = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.};$$

hoc igitur modo formula integralis proposita $\int \frac{dxlx}{\sqrt{1-xx}}$ transformatur in sequentem formam

$$\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.}\right),$$

ubi, cum sit $y=\sqrt{(1-xx)}$, notetur integrationem extendi debere ab $y=1$ usque ad $y=0$; quare si hos terminos integrationis permutare velimus, signum totius formae mutari oportet.

3. Quo autem minus tali signorum mutatione confundamur, designemus valorem quaesitum littera S, ut sit

$$S = \int \frac{dxlx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix},$$

atque facta substitutione $y=\sqrt{1-xx}$ habebimus, uti modo monuimus,

$$S = -\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \text{etc.}\right) \begin{bmatrix} \text{ab } y=0 \\ \text{ad } y=1 \end{bmatrix}.$$

Sub his autem integrationis terminis, scilicet ab $y=0$ ad $y=1$, iam satis notum est singulas partes, quae hic occurrunt, ad sequentes valores reduci:

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \frac{1}{2}\cdot\frac{\pi}{2},$$

$$\int \frac{y^4\,dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2},$$

$$\int \frac{y^6\,dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2},$$

$$\int \frac{y^8\,dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2},$$

$$\int \frac{y^{10}\,dy}{\sqrt{(1-yy)}} = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{\pi}{2}$$

etc.,

ubi nimirum est $\frac{\pi}{2} = \int \frac{dy}{\sqrt{(1-yy)}}$, ita ut $1:\pi$ exprimat rationem diametri ad peripheriam circuli.

4. Quodsi ergo singulos istos valores introducamus, pro valore quaesito S impetrabimus sequentem seriem infinitam

$$S = -\frac{\pi}{2}\left(\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2} + \text{etc.}\right)$$

sicque nunc totum negotium eo est reductum, ut istius seriei infinitae summa investigetur; qui labor fortasse haud minus operosus videri potest quam id ipsum, quod nobis exsequi est propositum. Interim tamen ad cognitionem summae huius seriei haud difficulter sequenti modo nobis pertingere licebit.

5. Cum sit

$$\frac{1}{\sqrt{(1-zz)}} = 1 + \frac{1}{2}zz + \frac{1\cdot 3}{2\cdot 4}z^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^6 + \text{etc.},$$

si utrinque per $\frac{dz}{z}$ multiplicemus et integremus, obtinebimus

$$\int \frac{dz}{z\sqrt{1-zz}} = lz + \frac{1}{2^2}zz + \frac{1\cdot 3}{2\cdot 4^2}z^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2}z^6 + \text{etc.}$$

sicque ad ipsam seriem nostram sumus perducti, cuius ergo valor quaeri debet ex hac expressione $\int \frac{dz}{z\sqrt{1-zz}} - lz$, integrali scilicet ita sumto, ut evanescat posito $z=0$; quo facto statuatur $z=1$ ac prodibit ipsa series

$$\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2} + \text{etc.}$$

Hoc igitur modo totum negotium perductum est ad istam formulam integralem $\int \frac{dz}{z\sqrt{1-zz}}$, quae posito $\sqrt{1-zz}=v$ transit in hanc formam $\frac{-dv}{1-vv}$, cuius integrale constat esse

$$-\frac{1}{2}l\frac{1+v}{1-v} = -l\frac{1+v}{\sqrt{(1-vv)}}.$$

Quodsi loco v restituatur valor $\sqrt{1-zz}$, tota expressio, qua indigemus, ita se habebit:

$$\int \frac{dz}{z\sqrt{(1-zz)}} - lz = -l\frac{1+\sqrt{1-zz}}{z} - lz + C = C - l\left(1+\sqrt{1-zz}\right),$$

ubi constans ita accipi debet, ut valor evanescat posito $z=0$, ideoque erit $C=l2$. Quamobrem posito $z=1$ summa seriei quaesita erit $l2$ hincque valor ipsius formulae integralis propositae erit

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} = S = -\frac{\pi}{2}l2,$$

prorsus uti longe alia methodo inveneram, ex quo iam satis intelligitur istam veritatem utique altioris esse indaginis ideoque attentione Geometrarum maxime dignam.

ALIA DEMONSTRATIO INTEGRATIONIS PROPOSITAE

6. Cum sit $\frac{dx}{\sqrt{1-xx}}$ elementum arcus circuli, cuius sinus $=x$, ponamus istum angulum $=\varphi$, ita ut sit $x=\sin.\varphi$ et $\frac{dx}{\sqrt{1-xx}}=d\varphi$, atque facta hac substitutione valor quantitatis S, in quem inquirimus, ita repraesentabitur

$$S = \int d\varphi\, l\sin.\varphi \begin{bmatrix}\text{a } \varphi=0\\ \text{ad } \varphi=90^0\end{bmatrix}.$$

Cum enim ante termini fuissent $x=0$ et $x=1$, iis nunc respondent $\varphi=0$ et $\varphi=90^0$ sive $\varphi=\frac{\pi}{2}$. Hic igitur totum negotium eo redit, ut formula $l\sin.\varphi$ commode in seriem infinitam convertatur. Hunc in finem ponamus $l\sin.\varphi=s$ eritque $ds=\frac{d\varphi\cos.\varphi}{\sin.\varphi}$. Novimus autem esse

$$\frac{\cos.\varphi}{\sin.\varphi}=2\sin.2\varphi+2\sin.4\varphi+2\sin.6\varphi+2\sin.8\varphi+\text{etc.}$$

Si enim utrinque per $\sin.\varphi$ multiplicemus, ob

$$2\sin.n\varphi\sin.\varphi=\cos.(n-1)\varphi-\cos.(n+1)\varphi$$

utique prodit

$$\begin{aligned}\cos.\varphi=\cos.\varphi&+\cos.3\varphi+\cos.5\varphi+\cos.7\varphi+\cos.9\varphi+\text{etc.}\\&-\cos.3\varphi-\cos.5\varphi-\cos.7\varphi-\cos.9\varphi-\text{etc.}\end{aligned}$$

Hac igitur serie pro $\frac{\cos.\varphi}{\sin.\varphi}$ in usum vocata erit

$$s=C-\cos.2\varphi-\frac{1}{2}\cos.4\varphi-\frac{1}{3}\cos.6\varphi-\frac{1}{4}\cos.8\varphi-\frac{1}{5}\cos.10\varphi-\text{etc.},$$

ubi, cum sit $s=l\sin.\varphi$ ideoque $s=0$, quando $\sin.\varphi=1$ ideoque $\varphi=\frac{\pi}{2}$, constantem C ita definire oportet, ut posito $\varphi=\frac{\pi}{2}=90^0$ evadat $s=0$, ex quo colligitur fore

$$C=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\text{etc.}=-l2.$$

7. Cum igitur sit

$$l\sin.\varphi=-l2-\cos.2\varphi-\frac{1}{2}\cos.4\varphi-\frac{1}{3}\cos.6\varphi-\frac{1}{4}\cos.8\varphi-\text{etc.},$$

erit valor formulae propositae

$$\begin{aligned}\int d\varphi\, l\sin.\varphi=C-\varphi l2-\frac{1}{2}\sin.2\varphi-\frac{1}{8}\sin.4\varphi-\frac{1}{18}\sin.6\varphi\\-\frac{1}{32}\sin.8\varphi-\frac{1}{50}\sin.10\varphi-\text{etc.};\end{aligned}$$

quae expressio cum evanescere debeat posito $\varphi = 0$, constans hic ingressa erit $C = 0$, ita ut iam in genere sit

$$\int d\varphi l \sin.\varphi = -\varphi l2 - \frac{2\sin.2\varphi}{2^2} - \frac{2\sin.4\varphi}{4^2} - \frac{2\sin.6\varphi}{6^2} - \frac{2\sin.8\varphi}{8^2} - \frac{2\sin.10\varphi}{10^2} - \frac{2\sin.12\varphi}{12^2} - \text{etc.}$$

Quodsi iam hic capiatur $\varphi = 90^0 = \frac{\pi}{2}$, omnium angulorum 2φ, 4φ, 6φ, 8φ etc., qui hic occurrunt, sinus evanescunt ideoque valor quaesitus erit

$$S = \int d\varphi l \sin.\varphi \begin{bmatrix} \text{a} & \varphi = 0 \\ \text{ad} & \varphi = 90^0 \end{bmatrix} = -\frac{\pi}{2} l2,$$

quemadmodum etiam in priore demonstratione ostendimus.

8. Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu, quo $\varphi = 90^0$, sed etiam verum eius valorem ostendat, quicunque angulus pro φ accipiatur, id quod ad ipsam formulam propositam $\int \frac{dx\,lx}{\sqrt{1-xx}}$ transferri poterit, cuius adeo valorem pro quolibet valore ipsius x assignare poterimus. Quodsi enim istius formulae valorem desideremus ab $x = 0$ usque ad $x = a$, quaeratur angulus α, cuius sinus sit aequalis ipsi a, atque semper habebitur

$$\int \frac{dx\,lx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = a \end{bmatrix} = -\alpha l2 - \frac{2\sin.2\alpha}{2^2} - \frac{2\sin.4\alpha}{4^2} - \frac{2\sin.6\alpha}{6^2} - \frac{2\sin.8\alpha}{8^2} - \text{etc.}$$

Unde patet, quoties fuerit $\alpha = \frac{i\pi}{2}$ denotante i numerum integrum quemcunque, quoniam omnes sinus evanescunt, valorem formulae his casibus finite exprimi per $-\frac{i\pi}{2} l2$; aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur. Ita si capiatur $a = \frac{1}{\sqrt{2}}$, ut sit $\alpha = \frac{\pi}{4}$, valor nostrae formulae erit

$$-\frac{\pi}{4} l2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.},$$

quae series elegantius ita exprimitur

$$-\frac{\pi}{4} l2 - \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \text{etc.}\right),$$

sicque hic occurrit series satis memorabilis

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\frac{1}{121}+\text{etc.},$$

cuius summam nullo adhuc modo ad mensuras cognitas revocare licuit.

9. Quoniam tam egregia series hic se quasi praeter exspectationem obtulit, etiam alios casus evolvamus notabiliores sumamusque $a=\frac{1}{2}$, ut sit $\alpha=30^0=\frac{\pi}{6}$, atque nostrae formulae hoc casu valor erit

$$-\frac{\pi}{6}\,l2-\frac{\sqrt{3}}{2^2}-\frac{\sqrt{3}}{4^2}+\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}-\frac{\sqrt{3}}{16^2}+\text{etc.},$$

quae expressio ita exhiberi potest

$$-\frac{\pi}{6}\,l2-\frac{\sqrt{3}}{4}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right),$$

in qua serie quadrata multiplorum ternarii deficiunt.

Sumamus nunc simili modo $a=\frac{\sqrt{3}}{2}$, ut sit $\alpha=60^0=\frac{\pi}{3}$, ac valor nostrae formulae hoc casu prodibit

$$-\frac{\pi}{3}\,l2-\frac{\sqrt{3}}{2^2}+\frac{\sqrt{3}}{4^2}-\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}+\frac{\sqrt{3}}{16^2}-\text{etc.}$$

sive hoc modo exprimetur

$$-\frac{\pi}{3}\,l2-\frac{\sqrt{3}}{4}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right).$$

ADHUC ALIA DEMONSTRATIO INTEGRATIONIS PROPOSITAE

10. Introducatur in formulam nostram angulus φ, cuius cosinus sit $=x$, sive sit $x=\cos.\varphi$ et formula nostra induet hanc formam $-\int d\varphi\, l\cos.\varphi$, quod integrale a $\varphi=90^0$ usque ad $\varphi=0$ erit extendendum. Quodsi autem hos terminos permutemus, valor S, quem quaerimus, ita exprimetur

$$S=\int d\varphi\, l\cos.\varphi\begin{bmatrix}\text{a} & \varphi=0\\ \text{ad} & \varphi=90^0\end{bmatrix}.$$

Ut hic $l\cos.\varphi$ in seriem idoneam convertamus, statuamus ut ante $s = l\cos.\varphi$ eritque $ds = -\frac{d\varphi \sin.\varphi}{\cos.\varphi}$. Constat autem per seriem esse

$$\frac{\sin.\varphi}{\cos.\varphi} = 2\sin.2\varphi - 2\sin.4\varphi + 2\sin.6\varphi - 2\sin.8\varphi + \text{etc.}$$

Cum enim in genere sit

$$2\sin.n\varphi\cos.\varphi = \sin.(n+1)\varphi + \sin.(n-1)\varphi,$$

si utrinque per $\cos.\varphi$ multiplicemus, orietur

$$\begin{aligned} \sin.\varphi = & \qquad\qquad\quad \sin.3\varphi - \sin.5\varphi + \sin.7\varphi - \sin.9\varphi + \text{etc.} \\ & + \sin.\varphi - \sin.3\varphi + \sin.5\varphi - \sin.7\varphi + \sin.9\varphi - \text{etc.}; \end{aligned}$$

quare cum sit $ds = -\frac{d\varphi \sin.\varphi}{\cos.\varphi}$, erit nunc

$$s = C + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \frac{\cos.10\varphi}{5} - \text{etc.}$$

Quia igitur est $s = l\cos.\varphi$, evidens est posito $\varphi = 0$ fieri debere $s = 0$, unde colligitur

$$C = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \text{etc.} = -l2,$$

sicque erit

$$l\cos.\varphi = -l2 + \frac{\cos.2\varphi}{1} - \frac{\cos.4\varphi}{2} + \frac{\cos.6\varphi}{3} - \frac{\cos.8\varphi}{4} + \text{etc.},$$

quae series ducta in $d\varphi$ et integrata praebet

$$S = \int d\varphi\, l\cos.\varphi = C - \varphi l2 + \frac{\sin.2\varphi}{2} - \frac{\sin.4\varphi}{8} + \frac{\sin.6\varphi}{18} - \frac{\sin.8\varphi}{32} + \frac{\sin.10\varphi}{50} - \text{etc.};$$

quae expressio quia sponte evanescit posito $\varphi = 0$, inde patet fore $C = 0$ sicque habebimus

$$\int d\varphi\, l\cos.\varphi = -\varphi l2 + \frac{1}{2}\left(\frac{\sin.2\varphi}{1} - \frac{\sin.4\varphi}{2^2} + \frac{\sin.6\varphi}{3^2} - \frac{\sin.8\varphi}{4^2} + \frac{\sin.10\varphi}{5^2} - \text{etc.}\right).$$

Sumto igitur $\varphi = \frac{\pi}{2} = 90^0$ oritur ut ante $S = -\frac{\pi}{2} l2$. Praeterea vero etiam hinc integrale ad quemvis terminum usque extendere licet.

11. Quodsi formulam posteriorem a praecedente subtrahamus, adipiscemur in genere hanc integrationem

$$\int d\varphi\, l\, \text{tang.}\, \varphi = -\sin. 2\varphi - \frac{1}{3^2}\sin. 6\varphi - \frac{1}{5^2}\sin. 10\varphi - \text{etc.},$$

unde patet hoc integrale evanescere casibus $\varphi = 90^0$ et in genere $\varphi = i\frac{\pi}{2}$. Postquam igitur istam integrationem triplici modo demonstravimus, ipsam analysin, quae me primum huc perduxit, hic dilucide sum expositurus.

ANALYSIS AD INTEGRATIONEM FORMULAE $\int \frac{dx\,lx}{\sqrt{1-xx}}$ ALIARUMQUE SIMILIUM PERDUCENS

12. Tota haec analysis innititur sequenti lemmate a me iam olim[1]) demonstrato:

Posito brevitatis gratia

$$(1-x^n)^{\frac{m-n}{n}} = X$$

si hinc duae formulae integrales formentur

$$\int Xx^{p-1}dx \quad \text{et} \quad \int Xx^{q-1}dx,$$

quae a termino $x=0$ usque ad terminum $x=1$ extendantur, ratio horum valorum sequenti modo ad productum ex infinitis factoribus conflatum reduci potest

$$\frac{\int Xx^{p-1}dx}{\int Xx^{q-1}dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.},$$

ubi scilicet singuli factores tam numeratoris quam denominatoris continuo eadem quantitate n augentur. Hic autem probe tenendum est veritatem istius lemmatis subsistere non posse, nisi singulae m, n, p et q denotent numeros positivos, quos tamen semper tanquam integros spectare licet.

1) Vide L. EULERI Commentationem 122 (indicis ENESTROEMIANI): *De productis ex infinitis factoribus ortis*, Comment. acad. sc. Petrop. 11 (1739), 1750, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. A. G.

13. Circa has duas formulas integrales a termino $x=0$ usque ad $x=1$ extensas duo casus imprimis seorsim notari merentur, quibus integratio actu succedit verusque valor absolute assignari potest.

Prior casus locum habet, si fuerit $p=n$, ita ut formula sit $\int Xx^{n-1}dx$. Posito enim $x^n=y$ fiet

$$X=(1-y)^{\frac{m-n}{n}} \quad \text{et} \quad x^{n-1}dx=\frac{1}{n}dy$$

sicque ista formula evadet $\frac{1}{n}\int dy(1-y)^{\frac{m-n}{n}}$ pariter a termino $y=0$ usque ad $y=1$ extendenda, quae porro posito $1-y=z$ abit in hanc formulam $-\frac{1}{n}\int z^{\frac{m-n}{n}}dz$ a termino $z=1$ usque ad $z=0$ extendendam; eius ergo integrale manifesto est $-\frac{1}{m}z^{\frac{m}{n}}+\frac{1}{m}$, unde facto $z=0$ valor erit $=\frac{1}{m}$. Consequenter pro casu $p=n$ habebimus

$$\int Xx^{n-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{m}$$

sicque, si fuerit vel $p=n$ vel $q=n$, integrale absolute innotescit.

14. Alter casus notatu dignus est, quo $p=n-m$, ita ut formula integranda sit $\int Xx^{n-m-1}dx$; tum enim si ponatur $x(1-x^n)^{\frac{-1}{n}}$ sive $\frac{x}{(1-x^n)^{\frac{1}{n}}}=y$, posito $x=0$ fiet $y=0$, at posito $x=1$ fiet $y=\infty$; tum autem erit

$$y^{n-m}=\frac{x^{n-m}}{(1-x^n)^{\frac{n-m}{n}}}=Xx^{n-m},$$

unde formula integranda erit $\int y^{n-m}\frac{dx}{x}$. Cum igitur sit $\frac{x}{(1-x^n)^{\frac{1}{n}}}=y$, erit $\frac{x^n}{1-x^n}=y^n$, unde colligitur $x^n=\frac{y^n}{1+y^n}$ ideoque $nlx=nly-l(1+y^n)$, cuius differentiatio praebet $\frac{dx}{x}=\frac{dy}{y(1+y^n)}$, quo valore substituto formula nostra integranda erit

$$\int\frac{y^{n-m-1}dy}{1+y^n}$$

a termino $y=0$ usque ad $y=\infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitate est liberata.

15. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat eius integrationem semper per logarithmos et arcus circulares absolvi posse; tum vero pro hoc casu non ita pridem[1]) ostendi huius formulae $\int \frac{x^{m-1}dx}{1+x^n}$ integrale ab $x=0$ usque ad $x=\infty$ extensum reduci ad valorem $\frac{\pi}{n\sin.\frac{m\pi}{n}}$. Facta igitur applicatione pro nostro casu habebimus

$$\int \frac{y^{n-m-1}dy}{1+y^n} = \frac{\pi}{n\sin.\frac{(n-m)\pi}{n}} = \frac{\pi}{n\sin.\frac{m\pi}{n}};$$

quamobrem pro casu $p=n-m$ valor integralis sequenti modo absolute exprimi potest eritque

$$\int Xx^{n-m-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{n\sin.\frac{m\pi}{n}},$$

quod idem manifesto tenendum est, si fuerit $q=n-m$.

16. His praemissis ponamus porro brevitatis gratia

$$\int Xx^{p-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = P \quad \text{et} \quad \int Xx^{q-1}dx \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = Q$$

atque lemma allatum nobis praebet hanc aequationem

$$\frac{P}{Q} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.}$$

Hinc igitur sumendis logarithmis deducimus

$$\begin{aligned} lP-lQ = {} & l(m+p)-lp+l(m+p+n)-l(p+n)+l(m+p+2n)-l(p+2n)+\text{etc.}\\ & +lq-l(m+q)+l(q+n)-l(m+q+n)\ +l(q+2n)-l(m+q+2n)+\text{etc.}\end{aligned}$$

haecque aequalitas semper locum habebit, quicunque valores litteris m, n, p et q tribuantur, dummodo fuerint positivi.

1) Vide *Institutionum calculi integralis* vol. I, § 351, Petropoli 1768; LEONHARDI EULERI *Opera omnia*, series I, vol. 11, p. 222. A. G.

17. Cum igitur haec aequalitas in genere subsistat, etiam veritati erit consentanea, quando quaepiam harum litterarum m, n, p et q infinite parum immutantur sive tanquam variabiles spectantur. Hanc ob rem consideremus solam quantitatem p tanquam variabilem, ita ut reliquae litterae m, n et q maneant constantes, ideoque etiam quantitas Q erit constans, dum altera P variabitur; ex quo differentiando nanciscemur hanc aequationem

$$\frac{dP}{P}=\frac{dp}{m+p}-\frac{dp}{p}+\frac{dp}{m+p+n}-\frac{dp}{p+n}+\frac{dp}{m+p+2n}-\frac{dp}{p+2n}$$
$$+\frac{dp}{m+p+3n}-\frac{dp}{p+3n}+\text{etc.},$$

ubi totum negotium eo redit, quemadmodum differentiale formulae P, quae est integralis, exprimi oporteat.

18. Cum igitur P sit formula integralis solam quantitatem x tanquam variabilem involvens, quandoquidem in eius integratione exponens p ut constans tractari debet, demum post integrationem ipsam quantitatem P tanquam functionem duarum variabilium x et p spectare licebit, unde quaestio huc redit, quomodo valorem hoc charactere $\left(\frac{dP}{dp}\right)$ exprimi solitum investigari oporteat; qui si indicetur littera Π, aequatio ante inventa hanc induet formam

$$\frac{\Pi}{P}=\frac{1}{m+p}-\frac{1}{p}+\frac{1}{m+p+n}-\frac{1}{p+n}+\frac{1}{m+p+2n}-\frac{1}{p+2n}+\text{etc.}$$

Hanc vero seriem infinitam haud difficulter ad expressionem finitam revocare licebit hoc modo. Ponatur

$$s=\frac{v^{m+p}}{m+p}-\frac{v^{p}}{p}+\frac{v^{m+p+n}}{m+p+n}-\frac{v^{p+n}}{p+n}+\frac{v^{m+p+2n}}{m+p+2n}-\frac{v^{p+2n}}{p+2n}+\text{etc.},$$

ita ut facto $v=1$ littera s nobis exhibeat valorem quaesitum $\frac{\Pi}{P}$; at vero differentiatio nobis dabit

$$\frac{ds}{dv}=v^{m+p-1}-v^{p-1}+v^{m+p+n-1}-v^{p+n-1}+v^{m+p+2n-1}-v^{p+2n-1}+\text{etc.},$$

cuius seriei infinitae summa manifesto est

$$\frac{v^{m+p-1}-v^{p-1}}{1-v^{n}}=\frac{v^{p-1}(v^{m}-1)}{1-v^{n}}.$$

Hinc igitur vicissim concludimus fore

$$s=\int\frac{v^{p-1}(v^m-1)\,dv}{1-v^n},$$

quae formula integralis a $v=0$ usque ad $v=1$ est extendenda; sicque habebimus

$$\frac{\Pi}{P}=\int\frac{v^{p-1}(v^m-1)\,dv}{1-v^n}\begin{bmatrix}\text{a } v=0\\ \text{ad } v=1\end{bmatrix}.$$

19. Ad valorem autem $\left(\frac{dP}{dp}\right)$, quem hic littera Π indicavimus, investigandum ex principiis calculi integralis ad functiones duarum variabilium applicati iam satis notum est differentiale formulae integralis $P=\int Xx^{p-1}dx$ ex sola variabilitate ipsius p oriundum obtineri, si formula post signum integrationis posita Xx^{p-1} ex sola variabilitate ipsius p differentietur atque elementum dp signo integrationis praefigatur; at vero quia X non continet p, hic ut constans tractari debet, potestatis vero x^{p-1} differentiale hinc natum erit $x^{p-1}dp\,lx$; quamobrem ex hac differentiatione orietur

$$dP=dp\int Xx^{p-1}dx\,lx,$$

ita ut tantum post signum integrationis factor lx accesserit, ex quo manifestum est fore

$$\Pi=\int Xx^{p-1}\,dx\,lx\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix};$$

hinc igitur sequens theorema generale constituere licebit.

THEOREMA GENERALE

20. *Posito brevitatis gratia* $X=(1-x^n)^{\frac{m-n}{n}}$ *si sequentes formulae integrales omnes a termino* $x=0$ *ad terminum* $x=1$ *extendantur, sequens aequalitas semper erit veritati consentanea*

$$\frac{\int Xx^{p-1}dx\,lx}{\int Xx^{p-1}dx}=\int\frac{x^{p-1}(x^m-1)\,dx}{1-x^n}.$$

Nihil enim obstabat, quominus loco v scriberemus x, quandoquidem isti valores tantum a terminis integrationis pendent.

21. Hoc igitur modo deducti sumus ad integrationem huiusmodi formularum $\int Xx^{p-1}dx\,lx$, in quibus quantitas logarithmica lx post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int Xx^{p-1}dx\,lx = \int Xx^{p-1}dx\cdot\int\frac{x^{p-1}(x^m-1)\,dx}{1-x^n},$$

integralibus scilicet ab $x=0$ ad $x=1$ extensis, ubi brevitatis gratia posuimus $(1-x^n)^{\frac{m-n}{n}} = X$. Hinc igitur pro binis casibus memorabilibus supra [§ 13—15] expositis bina theoremata particularia derivemus.

THEOREMA PARTICULARE 1 QUO $p=n$

22. Quoniam supra [§ 13] vidimus casu $p=n$ fieri $\int Xx^{n-1}dx = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem

$$\int Xx^{n-1}dx\,lx = \frac{1}{m}\int\frac{x^{n-1}(x^m-1)\,dx}{1-x^n},$$

dum scilicet ambo integralia ab $x=0$ ad $x=1$ extenduntur.

THEOREMA PARTICULARE 2 QUO $p=n-m$

23. Quoniam pro hoc casu, quo $p=n-m$, supra [§ 15] ostendimus esse

$$\int Xx^{n-m-1}dx = \frac{\pi}{n\sin.\frac{m\pi}{n}},$$

nunc deducimur ad sequentem integrationem maxime notatu dignam

$$\int Xx^{n-m-1}dx\,lx = \frac{\pi}{n\sin.\frac{m\pi}{n}}\int\frac{x^{n-m-1}(x^m-1)\,dx}{1-x^n},$$

siquidem haec ambo integralia ab $x=0$ usque ad $x=1$ extendantur; ubi meminisse oportet esse $X=(1-x^n)^{\frac{m-n}{n}}$.

24. Hic probe notetur theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet m, n et p, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet,

dummodo singulis valores positivi tribuantur, ita ut semper valor huius formulae integralis $\int X x^{p-1} dx\,lx$, quam ob factorem lx tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat; quae cum sint generalissima, operae pretium erit nonnullos casus speciales evolvere.

I. EVOLUTIO CASUS QUO $m=1$ ET $n=2$

25. Hoc igitur casu erit $X=\frac{1}{\sqrt{1-xx}}$, unde pro hoc casu theorema generale ita se habebit:

$$\int \frac{x^{p-1}dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^{p-1}dx}{\sqrt{1-xx}} \cdot \int \frac{x^{p-1}dx}{1+x},$$

siquidem singula haec integralia ab $x=0$ ad $x=1$ extendantur. Quoniam igitur hic tantum exponens p arbitrio nostro relinquitur, hinc sequentia exempla perlustremus.

EXEMPLUM 1 QUO $p=1$

26. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{dx\,lx}{\sqrt{1-xx}} = -\int \frac{dx}{\sqrt{1-xx}} \cdot \int \frac{dx}{1+x},$$

ubi integralibus ab $x=0$ ad $x=1$ extensis notum est fieri

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2} \quad \text{et} \quad \int \frac{dx}{1+x} = l2,$$

ita ut iam habeamus

$$\int \frac{dx\,lx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\pi}{2} l2,$$

quae est ea ipsa formula, quam initio huius dissertationis tractavimus et cuius veritatem iam triplici demonstratione corroboravimus.

27. Eundem valorem elicere licet ex theoremate particulari secundo, quo erat $p=n-m$, siquidem nunc ob $n=2$ et $m=1$ erit $p=1$; inde enim ob

$X=\frac{1}{\sqrt{1-xx}}$ istud theorema praebet

$$\int\frac{dx\,lx}{\sqrt{1-xx}}=\frac{\pi}{2\sin.\frac{\pi}{2}}\int-\frac{dx}{1+x}=-\frac{\pi}{2}l2.$$

EXEMPLUM 2 QUO $p=2$

28. Hoc igitur casu aequatio superior hanc induet formam

$$\int\frac{x\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{x\,dx}{\sqrt{1-xx}}\cdot\int\frac{x\,dx}{1+x}.$$

Iam vero integralibus ab $x=0$ ad $x=1$ extensis notum est fore

$$\int\frac{x\,dx}{\sqrt{1-xx}}=1 \quad \text{et} \quad \int\frac{x\,dx}{1+x}=1-l2,$$

ita ut habeamus

$$\int\frac{x\,dx\,lx}{\sqrt{1-xx}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l2-1.$$

29. Quoniam in hac formula integrale $\int\frac{x\,dx}{\sqrt{1-xx}}$ algebraice exhiberi potest, cum sit $=1-\sqrt{1-xx}$, valor quaesitus etiam per reductiones consuetas erui potest, cum sit

$$\int\frac{x\,dx\,lx}{\sqrt{1-xx}}=\left(1-\sqrt{1-xx}\right)lx-\int\frac{dx}{x}\left(1-\sqrt{1-xx}\right);$$

positoque $x=1$ erit

$$\int\frac{x\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{dx}{x}\left(1-\sqrt{1-xx}\right),$$

ad quam formam integrandam fiat

$$1-\sqrt{1-xx}=z,$$

unde colligitur $xx=2z-zz$, ergo $2lx=lz+l(2-z)$, sicque fiet

$$\frac{dx}{x}=\frac{dz(1-z)}{z(2-z)},$$

quibus valoribus substitutis erit

$$+\int\frac{dx}{x}\left(1-\sqrt{1-xx}\right)=+\int\frac{dz(1-z)}{2-z},$$

qui ergo valor erit $=C-z-l(2-z)$. Quia igitur posito $x=0$ fit $z=0$, constans erit $C=+l2$; facto igitur $x=1$, quia tum fit $z=1$, iste valor integralis erit $l2-1$, prorsus ut ante.

30. Eundem valorem suppeditat theorema prius supra allatum, quo erat $p=n=2$; inde enim statim fit $\int\frac{xdxlx}{\sqrt{1-xx}}=\int-\frac{xdx}{1+x}$. Ante autem vidimus esse $\int\frac{xdx}{1+x}=1-l2$, ita ut etiam hinc prodeat valor quaesitus $l2-1$.

EXEMPLUM 3 QUO $p=3$

31. Hoc igitur casu aequatio in theoremate generali allata hanc induet formam

$$\int\frac{xxdxlx}{\sqrt{1-xx}}=-\int\frac{xxdx}{\sqrt{1-xx}}\cdot\int\frac{xxdx}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int\frac{xxdx}{\sqrt{1-xx}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{2}\cdot\frac{\pi}{2};$$

at vero fractio spuria $\frac{xx}{1+x}$ resolvitur in has partes

$$x-1+\frac{1}{1+x},$$

unde erit

$$\int\frac{xxdx}{1+x}=\frac{1}{2}xx-x+l(1+x),$$

quod integrale iam evanescit posito $x=0$; facto ergo $x=1$ eius valor erit $=-\frac{1}{2}+l2$; quamobrem integrale, quod quaerimus, erit

$$\int\frac{xxdxlx}{\sqrt{1-xx}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\pi}{4}\left(l2-\frac{1}{2}\right).$$

EXEMPLUM 4 QUO $p=4$

32. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^3 dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^3 dx}{\sqrt{1-xx}} \cdot \int \frac{x^3 dx}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{x^3 dx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{2}{3};$$

tum vero fractio spuria $\frac{x^3}{1+x}$ resolvitur in has partes

$$xx - x + 1 - \frac{1}{x+1},$$

unde integrando fit

$$\int \frac{x^3 dx}{1+x} = \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x),$$

ex quo valor formulae erit $= \frac{5}{6} - l2$. His ergo valoribus substitutis adipiscimur hanc integrationem

$$\int \frac{x^3 dx\,lx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{2}{3}\left(\frac{5}{6} - l2\right).$$

EXEMPLUM 5 QUO $p=5$

33. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^4 dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^4 dx}{\sqrt{1-xx}} \cdot \int \frac{x^4 dx}{1+x}.$$

Constat autem esse

$$\int \frac{x^4 dx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2};$$

tum vero fractio spuria $\frac{x^4}{1+x}$ manifesto resolvitur in has partes

$$x^3 - xx + x - 1 + \frac{1}{x+1},$$

unde integrando fit

$$\int \frac{x^4 dx}{1+x} = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}xx - x + l(1+x),$$

ex quo valor formulae erit $= -\frac{7}{12} + l2$. His igitur valoribus substitutis prodibit ista integratio

$$\int \frac{x^4 dx\,lx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\left(l2 - \frac{7}{12}\right).$$

EXEMPLUM 6 QUO $p=6$

34. Hoc igitur casu aequatio superior induet hanc formam

$$\int \frac{x^5 dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^5 dx}{\sqrt{1-xx}}\cdot\int \frac{x^5 dx}{1+x}.$$

Constat autem per reductiones notas esse

$$\int \frac{x^5 dx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{2\cdot 4}{3\cdot 5};$$

tum vero fractio spuria $\frac{x^6}{1+x}$ resolvitur in has partes

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1},$$

unde integrando nanciscimur

$$\int \frac{x^5 dx}{1+x} = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x),$$

ex quo valor huius formulae erit $= \frac{47}{60} - l2$; quibus valoribus substitutis prodibit ista integratio

$$\int \frac{x^5 dx\,lx}{\sqrt{1-xx}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{2\cdot 4}{3\cdot 5}\left(\frac{47}{60} - l2\right).$$

II. EVOLUTIO CASUS QUO $m=3$ ET $n=2$

35. Hic ergo erit $X = \sqrt{(1-xx)}$, unde theorema nostrum generale nobis praebebit hanc aequationem

$$\int x^{p-1} dx\,lx\cdot\sqrt{1-xx} = \int x^{p-1} dx\sqrt{1-xx}\cdot\int \frac{x^{p-1}(x^3-1)dx}{1-xx};$$

ubi cum sit

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1},$$

erit postrema formula integralis

$$-\int x^p\,dx - \int \frac{x^{p-1}dx}{1+x},$$

quae integrata ab $x=0$ ad $x=1$ dat

$$-\frac{1}{p+1} - \int \frac{x^{p-1}dx}{1+x},$$

quamobrem habebimus

$$\int x^{p-1}\,dx lx \cdot \sqrt{1-xx} = -\int x^{p-1}\,dx \sqrt{1-xx} \cdot \left(\frac{1}{p+1} + \int \frac{x^{p-1}dx}{1+x}\right).$$

Hinc igitur sequentia exempla notasse iuvabit.

EXEMPLUM 1 QUO $p=1$

36. Pro hoc igitur casu postremus factor evadet $\frac{1}{2} + l2$, ita ut sit

$$\int dx lx \cdot \sqrt{1-xx} = -\left(\frac{1}{2} + l2\right) \int dx \sqrt{1-xx}.$$

Pro formula autem $\int dx \sqrt{1-xx}$ statuatur

$$\sqrt{1-xx} = 1 - vx$$

fietque $x = \frac{2v}{1+vv}$ et $\sqrt{1-xx} = \frac{1-vv}{1+vv}$ atque $dx = \frac{2dv(1-vv)}{(1+vv)^2}$, unde fiet

$$dx\sqrt{1-xx} = \frac{2dv(1-vv)^2}{(1+vv)^3},$$

cuius integrale resolvitur in has partes

$$\frac{2v}{(1+vv)^2} - \frac{v}{1+vv} + \text{A tang.}\, v;$$

quae expressio cum extendi debeat ab $x=0$ usque ad $x=1$, prior terminus erit $v=0$, alter vero terminus est $v=1$, ita ut integrale illud a

$v=0$ usque ad $v=1$ extendi debeat. At vero illa expressio sponte evanescit posito $v=0$, facto autem $v=1$ valor integralis erit $=\frac{\pi}{4}$; quamobrem habebimus

$$\int dx\,lx\cdot\sqrt{1-xx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\pi}{4}\left(\frac{1}{2}+l2\right).$$

37. Hic quidem calculum per longas ambages evolvimus, prouti reductio ad rationalitatem formulae $\sqrt{1-xx}$ manuduxit; at vero solus aspectus formulae $\int dx\sqrt{1-xx}$ statim declarat eam exprimere aream quadrantis circuli, cuius radius $=1$, quem novimus esse $=\frac{\pi}{4}$. Caeterum adhiberi potuisset ista reductio

$$\int dx\sqrt{1-xx}=\frac{1}{2}x\sqrt{1-xx}+\frac{1}{2}\int\frac{dx}{\sqrt{1-xx}},$$

cuius valor ab $x=0$ ad $x=1$ extensus manifesto dat $\frac{\pi}{4}$.

EXEMPLUM 2 QUO $p=2$

38. Hoc ergo casu postremus factor fit

$$\frac{1}{3}+\int\frac{xdx}{1+x}=\frac{4}{3}-l2$$

sicque habebimus

$$\int xdx\,lx\cdot\sqrt{1-xx}=-\left(\frac{4}{3}-l2\right)\int xdx\sqrt{1-xx};$$

perspicuum autem est esse

$$\int xdx\sqrt{1-xx}=C-\frac{1}{3}(1-xx)^{\frac{3}{2}},$$

qui valor ab $x=0$ ad $x=1$ extensus praebet $\frac{1}{3}$, ita ut habeamus

$$\int xdx\,lx\cdot\sqrt{1-xx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{1}{3}\left(\frac{4}{3}-l2\right).$$

III. EVOLUTIO CASUS QUO $m=1$ ET $n=3$

39. Hoc igitur casu erit $X=\frac{1}{\sqrt[3]{(1-x^3)^2}}$, unde theorema generale nobis praebet hanc aequationem

$$\int\frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)^2}}=\int\frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x^{p-1}(x-1)\,dx}{1-x^3},$$

ubi postrema formula reducitur ad hanc $-\int\frac{x^{p-1}dx}{xx+x+1}$, ita ut habeamus

$$\int\frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)^2}}=-\int\frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x^{p-1}dx}{xx+x+1}.$$

Sequentia igitur exempla adiungamus.

EXEMPLUM 1 QUO $p=1$

40. Hoc igitur casu postremus factor evadit $\int\frac{dx}{xx+x+1}$, cuius integrale indefinitum reperitur $\frac{2}{\sqrt{3}}\,\text{A tang.}\,\frac{x\sqrt{3}}{2+x}$, qui valor posito $x=1$ abit in $\frac{\pi}{3\sqrt{3}}$; quocirca hoc casu habebimus

$$\int\frac{dx\,lx}{\sqrt[3]{(1-x^3)^2}}=-\frac{\pi}{3\sqrt{3}}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}};$$

at vero formula integralis $\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}$ peculiarem quantitatem transcendentem involvit, quam neque per logarithmos neque per arcus circulares explicare licet.

EXEMPLUM 2 QUO $p=2$

41. Hoc igitur casu postremus factor erit $\int\frac{x\,dx}{1+x+xx}$, qui in has partes resolvatur

$$\frac{1}{2}\int\frac{2x\,dx+dx}{1+x+xx}-\frac{1}{2}\int\frac{dx}{1+x+xx},$$

ubi partis prioris integrale est

$$\frac{1}{2}l(1+x+xx)=\frac{1}{2}l3\ \text{(posito scilicet } x=1),$$

alterius vero partis integrale est $-\frac{1}{2}\cdot\frac{\pi}{3\sqrt{3}}$; quo valore substituto habebimus

$$\int \frac{x\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} = -\frac{1}{2}\left(l3-\frac{\pi}{3\sqrt{3}}\right)\int \frac{x\,dx}{\sqrt[3]{(1-x^3)^2}}.$$

Nunc vero istam formulam integralem commode assignare licet per reductionem supra initio indicatam; cum enim hic sit $m=1$ et $n=3$, tum vero sumserimus $p=2$, erit $p=n-m$. Supra autem (§ 15) invenimus hoc casu integrale fore $=\frac{\pi}{n\sin.\frac{m\pi}{n}}$, qui valor nostro casu abit in $\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}$. Hoc igitur valore substituto nostram formulam per meras quantitates cognitas exprimere poterimus hoc modo

$$\int \frac{x\,dx\,lx}{\sqrt[3]{(1-x^3)^2}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{\pi}{3\sqrt{3}}\left(l3-\frac{\pi}{3\sqrt{3}}\right).$$

IV. EVOLUTIO CASUS QUO $m=2$ ET $n=3$

42. Hoc igitur casu erit $X=\frac{1}{\sqrt[3]{(1-x^3)}}$, unde theorema generale praebet istam aequationem

$$\int \frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)}} = \int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int \frac{x^{p-1}(xx-1)dx}{1-x^3},$$

ubi forma postrema transmutatur in hanc $-\int \frac{x^{p-1}dx(1+x)}{1+x+xx}$; unde fiet

$$\int \frac{x^{p-1}dx\,lx}{\sqrt[3]{(1-x^3)}} = -\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int \frac{x^{p-1}dx(1+x)}{1+x+xx},$$

unde sequentia exempla expediamus.

EXEMPLUM 1 QUO $p=1$

43. Hoc ergo casu membrum postremum erit $\int \frac{dx(1+x)}{1+x+xx}$, cuius integrale in has partes distribuatur

$$\frac{1}{2}\int \frac{2x\,dx+dx}{1+x+xx}+\frac{1}{2}\int \frac{dx}{1+x+xx},$$

unde manifesto pro casu $x=1$ prodit $\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)$; quamobrem nostra aequatio erit

$$\int \frac{dx lx}{\sqrt[3]{1-x^3}} = -\frac{1}{2}\left(l3+\frac{\pi}{3\sqrt{3}}\right)\int \frac{dx}{\sqrt[3]{(1-x^3)}}.$$

In hac autem formula integrali ob $m=2$ et $n=3$, quia sumsimus $p=1$, erit $p=n-m$; pro hoc ergo casu per § 15 valor istius formulae absolute exprimi poterit eritque $\int \frac{dx}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}}$; consequenter etiam hoc casu per quantitates absolutas consequimur hanc formam

$$\int \frac{dx lx}{\sqrt[3]{(1-x^3)}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = -\frac{\pi}{3\sqrt{3}}\left(l3+\frac{\pi}{3\sqrt{3}}\right).$$

44. Quodsi hanc formam cum postrema casus praecedentis, quae itidem absolute prodiit expressa, combinemus, earum summa primo dabit

$$\int \frac{x dx lx}{\sqrt[3]{(1-x^3)^2}} + \int \frac{dx lx}{\sqrt[3]{(1-x^3)}} = -\frac{2\pi l3}{3\sqrt{3}},$$

sin autem posterior a priore subtrahatur, orietur ista aequatio

$$\int \frac{x dx lx}{\sqrt[3]{(1-x^3)^2}} - \int \frac{dx lx}{\sqrt[3]{(1-x^3)}} = \frac{2\pi\pi}{27}.$$

Quoniam hoc modo ad expressiones satis simplices sumus perducti, operae pretium erit ambas aequationes sub alia forma repraesentare, qua binae partes integrales commode in unam coniungi queant; statuamus scilicet

$$\frac{x}{\sqrt[3]{(1-x^3)}} = z,$$

unde fit $\frac{xx}{\sqrt[3]{(1-x^3)^2}} = zz$, sicque prior formula induet hanc speciem $\int \frac{zz dx lx}{x}$, posterior vero istam $\int \frac{z dx lx}{x}$; tum vero habebimus $\frac{x^3}{1-x^3} = z^3$, unde fit $x^3 = \frac{z^3}{1+z^3}$ ideoque

$$lx = lz - \frac{1}{3}l(1+z^3) = l\frac{z}{\sqrt[3]{(1+z^3)}}$$

hincque porro

$$\frac{dx}{x} = \frac{dz}{z} - \frac{zzdz}{1+z^3} = \frac{dz}{z(1+z^3)},$$

quare his valoribus adhibitis prior formula integralis evadit $\int \frac{zdz}{1+z^3} l \frac{z}{\sqrt[3]{(1+z^3)}}$, altera vero formula erit $\int \frac{dz}{1+z^3} l \frac{z}{\sqrt[3]{(1+z^3)}}$.

45. Quoniam autem integralia ab $x=0$ ad $x=1$ extendi debent, notandum est casu $x=0$ fieri $z=0$, at vero casu $x=1$ prodire $z=\infty$, ita ut novas istas formas a $z=0$ ad $z=\infty$ extendi oporteat. Quo observato prior harum formularum dabit

$$\int \frac{zdz}{1+z^3} l \frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} \text{a} & z=0 \\ \text{ad} & z=\infty \end{bmatrix} = -\frac{\pi l3}{3\sqrt{3}} + \frac{\pi\pi}{27},$$

posterior vero

$$\int \frac{dz}{1+z^3} l \frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} \text{a} & z=0 \\ \text{ad} & z=\infty \end{bmatrix} = -\frac{\pi l3}{3\sqrt{3}} - \frac{\pi\pi}{27}.$$

Hinc igitur summa harum formularum erit

$$\int \frac{dz(1+z)}{1+z^3} l \frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{2\pi l3}{3\sqrt{3}},$$

at vero differentia

$$\int \frac{dz(z-1)}{1+z^3} l \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{2\pi\pi}{27}.$$

46. Hic non inutile erit observasse istum logarithmum $l \frac{z}{\sqrt[3]{(1+z^3)}}$ commode in seriem infinitam satis simplicem converti posse; cum enim sit

$$l \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{1}{3} l \frac{z^3}{1+z^3} = -\frac{1}{3} l \frac{1+z^3}{z^3},$$

erit per seriem

$$l \frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{1}{3}\left(\frac{1}{z^3} - \frac{1}{2z^6} + \frac{1}{3z^9} - \frac{1}{4z^{12}} + \frac{1}{5z^{15}} - \text{etc.}\right);$$

verum ista resolutio nullum usum praestare potest ad integralia haec per series evolvenda, propterea quod potestates ipsius z in denominatoribus occurrunt ideoque singulae partes non ita integrari possunt, ut evanescant posito $z=0$.

EXEMPLUM 2 QUO $p=2$

47. Hoc igitur casu factor postremus evadit $\int \frac{x\,dx(1+x)}{1+x+xx}$, qui in has duas partes discerpitur

$$\int dx - \int \frac{dx}{1+x+xx},$$

cuius ergo integrale ab $x=0$ ad $x=1$ extensum est $=1-\frac{\pi}{3\sqrt{3}}$. Hinc igitur deducimur ad hanc aequationem

$$\int \frac{x\,dx\,lx}{\sqrt[3]{(1-x^3)}} = -\left(1-\frac{\pi}{3\sqrt{3}}\right)\int \frac{x\,dx}{\sqrt[3]{(1-x^3)}}.$$

Hic autem notandum est istam formulam integralem nullo modo absolute exhiberi posse, sed peculiarem quandam quantitatem transcendentem involvere.

V. EVOLUTIO CASUS QUO $m=2$ ET $n=4$

48. Hoc igitur casu erit $X=\frac{1}{\sqrt{(1-x^4)}}$, unde theorema nostrum generale nobis dabit hanc aequationem

$$\int \frac{x^{p-1}dx\,lx}{\sqrt{(1-x^4)}} = -\int \frac{x^{p-1}dx}{\sqrt{(1-x^4)}} \cdot \int \frac{x^{p-1}dx}{1+xx};$$

at vero problema particulare prius pro hoc casu praebet

$$\int \frac{x^3dx\,lx}{\sqrt{(1-x^4)}} = -\frac{1}{2}\int \frac{x^3dx}{1+xx}.$$

Cum autem sit

$$\int \frac{x^3dx}{1+xx} = \frac{1}{2}-\frac{1}{2}l2,$$

erit absolute

$$\int \frac{x^3dx\,lx}{\sqrt{(1-x^4)}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{1}{4}(1-l2),$$

at vero hic casus congruit cum supra (§ 28) tractato. Si enim hic ponamus $xx=y$, quo facto termini integrationis manent $y=0$ et $y=1$, erit $lx=\frac{1}{2}ly$ et $xdx=\frac{1}{2}dy$; quibus valoribus substitutis nostra aequatio abibit in hanc formam $\frac{1}{4}\int \frac{y\,dy\,ly}{\sqrt{(1-yy)}} = -\frac{1}{4}(1-l2)$ sive $\int \frac{y\,dy\,ly}{\sqrt{(1-yy)}} = l2-1$, prorsus ut supra.

49. Alterum vero theorema particulare ad praesentem casum accommodatum dabit

$$\int\frac{xdxlx}{\sqrt{(1-x^4)}}=-\frac{\pi}{4}\int\frac{xdx}{1+xx};$$

est vero

$$\int\frac{xdx}{1+xx}=l\sqrt{1+xx}=\frac{1}{2}l2,$$

ita ut habeamus

$$\int\frac{xdxlx}{\sqrt{(1-x^4)}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\pi}{8}l2.$$

Quodsi vero hic ut ante statuamus $xx=y$, obtinebitur $\int\frac{dyly}{\sqrt{(1-yy)}}=-\frac{\pi}{2}l2$, qui est casus supra (§ 26) tractatus. His duobus casibus exponens p erat numerus par, unde casus impares evolvi conveniet.

EXEMPLUM 1 QUO $p=1$

50. Hoc igitur casu formula integralis postrema fiet

$$\int\frac{dx}{1+xx}=\text{A tang. }x,$$

ita ut posito $x=1$ prodeat $\frac{\pi}{4}$; tum vero aequatio nostra erit

$$\int\frac{dxlx}{\sqrt{(1-x^4)}}=-\frac{\pi}{4}\int\frac{dx}{\sqrt{(1-x^4)}},$$

integralibus scilicet ab $x=0$ ad $x=1$ extensis; ubi formula $\int\frac{dx}{\sqrt{(1-x^4)}}$ arcum curvae elasticae rectangulae exprimit ideoque absolute exhiberi nequit.

EXEMPLUM 2 QUO $p=3$

51. Hoc ergo casu formula integralis postrema erit

$$\int\frac{xxdx}{1+xx}=\int dx-\int\frac{dx}{1+xx},$$

cuius integrale posito $x=1$ fit $=1-\frac{\pi}{4}$, ita ut nunc aequatio nostra evadat

$$\int\frac{xxdxlx}{\sqrt{(1-x^4)}}=-\left(1-\frac{\pi}{4}\right)\int\frac{xxdx}{\sqrt{(1-x^4)}},$$

quae formula integralis pariter absolute exhiberi nequit; exprimit enim applicatam curvae elasticae rectangulae.

52. Quanquam autem haec duo exempla ad formulas inextricabiles perduxerunt, tamen iam pridem[1]) demonstravi productum horum duorum integralium $\int \frac{dx}{\sqrt{(1-x^4)}} \cdot \int \frac{xxdx}{\sqrt{(1-x^4)}}$ aequari areae circuli, cuius diameter $=1$, sive esse $=\frac{\pi}{4}$; quamobrem binis exemplis coniungendis hoc insigne theorema adipiscimur

$$\int \frac{dx\,lx}{\sqrt{(1-x^4)}} \cdot \int \frac{xxdx\,lx}{\sqrt{(1-x^4)}} = \frac{\pi^2}{16}\left(1-\frac{\pi}{4}\right).$$

Facile autem patet innumera alia huiusmodi theoremata ex hoc fonte hauriri posse, quae per se spectata profundissimae indaginis sunt censenda.

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1) Vide L. EULERI Commentationes 59 et 122 (indicis ENESTROEMIANI): *Theoremata circa reductionem formularum integralium ad quadraturam circuli,* Miscellanea Berolin. 7, 1743, p. 91, et *De productis ex infinitis factoribus ortis,* Comment. acad. sc. Petrop. **11** (1739), 1750, p. 3; *LEONHARDI EULERI Opera omnia,* series I, vol. 17 et 14. Vide etiam epistolam ab EULERO d. 20. Dec. 1738 ad IOH. BERNOULLI datam, Biblioth. Mathem. 5_3, 1904, p. 285, imprimis p. 291, atque epistolam, quam EULERUS d. 9. Sept. 1741 ad CHR. GOLDBACH misit, *Correspondance math. et phys. publiée par P. H. Fuss,* St. Pétersbourg 1843, t. I, p. 105, imprimis p. 107; *LEONHARDI EULERI Opera omnia,* series III. A. G.

DE VALORE FORMULAE INTEGRALIS

$$\int \frac{x^{a-1}dx}{lx} \cdot \frac{(1-x^b)(1-x^c)}{1-x^n}$$

A TERMINO $x=0$ USQUE AD $x=1$ EXTENSAE

Commentatio 500 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1777: II (1780), p. 29—47

1. Quae non ita pridem de integratione eiusmodi formularum differentialium, in quarum denominatore occurrit lx, in medium attuli, ubi ostendi[1]) valorem huius formulae integralis $\int \frac{x^{a-1}-x^{b-1}}{lx}dx$ ab $x=0$ ad $x=1$ extensae esse $=l\frac{a}{b}$, non solum summa attentione digna, sed etiam quasi novum campum in methodo integrandi aperire sunt visa, propterea quod huiusmodi formularum integratio prorsus singularia artificia postulat, at ex principiis etiamnunc parum cognitis erat deducta. Tunc quidem temporis ista investigatio non admodum late patere videbatur, dum praeter formulam modo allegatam ad paucas alias eam mihi quidem extendere licuit; nunc autem, postquam hoc argumentum accuratius sum perscrutatus, deprehendi formulam multo generaliorem, eam scilicet, quae hic in titulo conspicitur, pari successu expediri posse. Quin etiam methodus, quam hic sum expositurus, etiam ad formulas adhuc generaliores facile extendi potest, unde haud contemnenda incrementa in universam Analysin redundare videntur.

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1) Vide § 6 Commentationis 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. **19** (1774), 1775, p. 66; LEONHARDI EULERI *Opera omnia*, series I, vol. 17, p. 421, imprimis p. 426. A. G.

2. Designemus igitur littera S valorem formulae propositae, quem scilicet induit, si eius integratio a termino $x=0$ usque ad $x=1$ extendatur, ita ut sit

$$S=\int \frac{x^{a-1}dx}{lx}\cdot\frac{(1-x^b)(1-x^c)}{1-x^n}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix},$$

ad quem valorem investigandum ante omnia observari convenit fractionem $\frac{(1-x^b)(1-x^c)}{1-x^n}$ ita esse comparatam, ut posito $x=1$ penitus evanescat. Cum enim in numeratore tam $1-x^b$ quam $1-x^c$ factorem $1-x$ involvat ideoque totus numerator factorem habeat $(1-x)^2$, dum in denominatore tantum factor simplex $1-x$ inest, evidens est posito $x=1$ totam fractionem evanescere debere; id quod etiam inde intelligitur, quod casu $x=1$ tam numerator quam denominator evanescit, unde, si iuxta regulam notissimam tam loco numeratoris, qui evolutus est $1-x^b-x^c+x^{b+c}$, quam loco denominatoris utriusque differentialia scribantur, prodit ista fractio

$$\frac{-bx^{b-1}-cx^{c-1}+(b+c)x^{b+c-1}}{-nx^{n-1}}$$

illi aequalis casu $x=1$; posito autem $x=1$ ista fractio abit in hanc $\frac{-b-c+b+c}{-n}$, quae manifesto est $=0$.

3. Cum numerator fractionis modo consideratae sit $1-x^b-x^c+x^{b+c}$, si is per $1-x^n$ dividatur, ex quaternis terminis orientur quatuor sequentes series geometricae infinitae

I. $1+x^n+x^{2n}+x^{3n}+x^{4n}+x^{5n}+\text{etc.},$

II. $-x^b-x^{n+b}-x^{2n+b}-x^{3n+b}-x^{4n+b}-x^{5n+b}-\text{etc.},$

III. $-x^c-x^{n+c}-x^{2n+c}-x^{3n+c}-x^{4n+c}-x^{5n+c}-\text{etc.},$

IV. $x^{b+c}+x^{n+b+c}+x^{2n+b+c}+x^{3n+b+c}+x^{4n+b+c}+x^{5n+b+c}+\text{etc.}$

Harum igitur serierum singulos terminos duci oportet in formulam $\frac{x^{a-1}dx}{lx}$; tum enim omnium integralia ab $x=0$ ad $x=1$ extensa, si in unam summam colligantur, dabunt valorem quaesitum littera S designatum.

4. Hoc ergo modo totum negotium reducitur ad integrationem talis formulae $\frac{x^m\,dx}{lx}$ ab $x=0$ ad $x=1$ extendendam. Haec autem formula continet fundamentum principale, unde omnia, quae olim[1]) de hoc argumento sum commentatus, sunt deducta; tum autem ad eius integrale inveniendum usus sum doctrina circa functiones duarum variabilium versante, quam ad praesens institutum non satis commode applicare liceret; quamobrem hic aliam methodum in medium sum allaturus, cuius beneficio ista integratio, qua indigemus, multo facilius et clarius institui poterit et qua simul omnia, quae huc pertinent, haud mediocriter illustrabuntur.

5. Cum sit $lx^m = mlx$, si littera e denotet numerum, cuius logarithmus hyperbolicus unitati aequatur, posito brevitatis gratia $mlx = y$ erit $lx^m = y = yle$ hincque vicissim fiet $x^m = e^y = e^{mlx}$. Cum igitur per seriem notissimam sit

$$e^y = 1 + \frac{y}{1} + \frac{yy}{1\cdot 2} + \frac{y^3}{1\cdot 2\cdot 3} + \frac{y^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

erit pro nostro casu

$$x^m = 1 + \frac{mlx}{1} + \frac{mm}{1\cdot 2}(lx)^2 + \frac{m^3}{1\cdot 2\cdot 3}(lx)^3 + \frac{m^4}{1\cdot 2\cdot 3\cdot 4}(lx)^4 + \text{etc.};$$

hac igitur serie in usum vocata erit

$$\frac{x^m}{lx} = \frac{1}{lx} + \frac{m}{1} + \frac{mm}{1\cdot 2}lx + \frac{m^3}{1\cdot 2\cdot 3}(lx)^2 + \frac{m^4}{1\cdot 2\cdot 3\cdot 4}(lx)^3 + \frac{m^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}(lx)^4 + \text{etc.}$$

Huius igitur seriei singulos terminos in dx ductos integrari oportet, unde quidem ex termino primo orietur formula $\int\frac{dx}{lx}$, cuius valorem ab $x=0$ ad $x=1$ extensum esse infinitum ostendi[2]), cuius loco hic ubique scribamus characterem Δ; tum vero ex termino secundo oritur integrale $\frac{m}{1}x = m$.

1) Vide Commentationem 463 (indicis ENESTROEMIANI): *De valore formulae integralis* $\int\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}}\cdot\frac{dz}{z}(lz)^\mu$ *casu, quo post integrationem ponitur* $z=1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 30; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 384; vide etiam Commentationem 464 supra (nota p. 51) laudatam. A. G.

2) Vide *Institutionum calculi integralis* vol. I, § 228, Petropoli 1768; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 127. A. G.

6. Pro integralibus ex reliquis terminis oriundis ex elementis calculi integralis satis liquet, si integralia ab $x=0$ ad $x=1$ extendantur, fore ut sequitur:

$$\int dx\,lx = -1,\quad \int dx(lx)^2 = +1\cdot 2,\quad \int dx(lx)^3 = -1\cdot 2\cdot 3,$$

$$\int dx(lx)^4 = +1\cdot 2\cdot 3\cdot 4,\quad \int dx(lx)^5 = -1\cdot 2\cdot 3\cdot 4\cdot 5 \quad \text{etc.};$$

his igitur valoribus substitutis reperiemus fore

$$\int \frac{x^m\,dx}{lx} = \varDelta + m - \frac{mm}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \frac{m^5}{5} - \frac{m^6}{6} + \frac{m^7}{7} - \text{etc.}$$

Ex doctrina autem logarithmorum constat esse

$$l(1+m) = m - \frac{mm}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \text{etc.},$$

quo valore substituto habebimus

$$\int \frac{x^m\,dx}{lx} = \varDelta + l(1+m),$$

qui ergo est valor huius formulae integralis a termino $x=0$ ad $x=1$ extensae, quos terminos in sequentibus semper subintelligi oportet, unde eos non amplius commemorabimus.

7. Iste quidem valor integralis insigni incommodo laborare videtur, propterea quod characterem $\varDelta$ implicat, cuius valor non solum est incognitus, sed adeo infinitus; verum quia pro omnibus huiusmodi formulis perpetuo idem manet, ita ut sit

$$\int \frac{x^n\,dx}{lx} = \varDelta + l(1+n),$$

evidens est, si harum formularum altera ab altera subtrahatur, istum characterem penitus ex calculo egredi ac prodire

$$\int \frac{x^m - x^n}{lx}\,dx = l\,\frac{1+m}{1+n},$$

qui est ille ipse casus, ad quem primo initio sum perductus. Quo autem

clarius appareat, quibusnam casibus iste character Δ penitus ex calculo sit excessurus, contemplemur hanc formam indefinitam

$$X = Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + Dx^{\delta} + Ex^{\varepsilon} + \text{etc.}$$

ac per integrale illud inventum erit

$$\int \frac{X\,dx}{lx} = A\Delta + B\Delta + C\Delta + D\Delta + \text{etc.}$$
$$+ Al(1+\alpha) + Bl(1+\beta) + Cl(1+\gamma) + Dl(1+\delta) + \text{etc.},$$

Quocirca si coefficientes A, B, C, D etc. ita fuerint comparati, ut sit $A + B + C + D + \text{etc.} = 0$, semper istud integrale ita exprimetur

$$\int \frac{X\,dx}{lx} = Al(1+\alpha) + Bl(1+\beta) + Cl(1+\gamma) + Dl(1+\delta) + \text{etc.},$$

perinde ac si formula canonica fuisset $\int \frac{x^m\,dx}{lx} = l(1+m)$ reiecto charactere Δ.

8. Quoties igitur fuerit

$$X = Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + Dx^{\delta} + \text{etc.}$$

existente $A + B + C + D + \text{etc.} = 0$, tum integrale $\int \frac{X\,dx}{lx}$ non amplius charactere Δ inquinabitur atque singulas integrationes ita instituere licebit, quasi revera foret

$$\int \frac{x^m\,dx}{lx} = l(1+m).$$

Cum igitur series $A + B + C + D + \text{etc.}$ exhibeat valorem ipsius X, si ponatur $x = 1$, manifestum est istam integrationem perpetuo succedere, si X eiusmodi exprimat functionem ipsius x, ut posito $x = 1$ ea in nihilum abeat. Quare cum formula, quam hic tractare suscepimus,

$$X = \frac{x^{a-1}(1-x^b)(1-x^c)}{1-x^n},$$

uti iam observavimus, ad nihilum redigitur posito $x = 1$, eius integrationem rite absolvere licebit ope formulae canonicae $\int \frac{x^m\,dx}{lx} = l(1+m)$, nullo scilicet respectu habito ad characterem Δ initio introductum.

9. Quoniam igitur iam supra perducti sumus ad quatuor series infinitas, quas per formulam $\frac{x^{n-1}dx}{lx}$ multiplicari, tum vero integrari oportet, si hanc operationem in singulis terminis instituamus, valor quaesitus S per sequentes quatuor series infinitas expressus reperietur:

$$S = \begin{cases} \text{I.} & la + l(a+n) + l(a+2n) + l(a+3n) + l(a+4n) + \text{etc.} \\ \text{II.} & -l(a+b) - l(a+b+n) - l(a+b+2n) - l(a+b+3n) \\ & \qquad - l(a+b+4n) - \text{etc.} \\ \text{III.} & -l(a+c) - l(a+c+n) - l(a+c+2n) - l(a+c+3n) \\ & \qquad - l(a+c+4n) - \text{etc.} \\ \text{IV.} & l(a+b+c) + l(a+b+c+n) + l(a+b+c+2n) + l(a+b+c+3n) \\ & \qquad + l(a+b+c+4n) + \text{etc.} \end{cases}$$

Hoc igitur modo tota quaestio huc est reducta, ut expressiones finitae investigentur, quae istis logarithmorum seriebus infinitis sint aequales.

10. Cum igitur valor quaesitus S infinitis logarithmis aequalis sit inventus, eum ipsum tanquam logarithmum spectari conveniet; quamobrem statuamus $S = lO$ atque a logarithmis ad numeros regrediendo valor ipsius O sequenti modo per factores exprimi deprehendetur

$$O = \frac{a(a+b+c)}{(a+b)(a+c)} \cdot \frac{(a+n)(a+b+c+n)}{(a+b+n)(a+c+n)} \cdot \frac{(a+2n)(a+b+c+2n)}{(a+b+2n)(a+c+2n)} \cdot \frac{(a+3n)(a+b+c+3n)}{(a+b+3n)(a+c+3n)} \cdot \text{etc.},$$

quam expressionem in membra puncto separata distinximus, quorum quodlibet continet binos factores in numeratore totidemque in denominatore, qui factores in singulis membris ita sunt comparati, ut summa factorum numeratoris semper aequalis sit summae factorum denominatoris. Praeterea vero notetur sumendo i pro numero infinito membrum infinitesimum esse

$$\frac{(a+in)(a+b+c+in)}{(a+b+in)(a+c+in)},$$

quod evolutum praebet

$$\frac{a(a+b+c) + in(2a+b+c) + iinn}{(a+b)(a+c) + in(2a+b+c) + iinn},$$

cuius valor ob partes primas finitas evanescentes manifesto unitati aequatur;

unde intelligitur hanc expressionem valorem finitum ac determinatum esse habituram, et quo plura membra actu in se invicem ducantur, eo propius continuo ad valorem ipsius O appropinquatum iri, quandoquidem membra satis remota continuo minus ab unitate discrepant.

11. Ut nunc in verum valorem litterae O inquiramus, in subsidium vocemus insigne lemma, cuius veritatem iam in *Calculo integrali*[1]) fusius demonstravi, quod ita se habet. Si ponatur

$$P=\int x^{p-1}dx(1-x^n)^{\frac{m-n}{n}} \quad \text{et} \quad Q=\int x^{q-1}dx(1-x^n)^{\frac{m-n}{n}},$$

tum erit

$$\frac{P}{Q}=\frac{(m+p)q}{p(m+q)}\cdot\frac{(m+p+n)(q+n)}{(p+n)(m+q+n)}\cdot\frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)}\cdot\frac{(m+p+3n)(q+3n)}{(p+3n)(m+q+3n)}\cdot\text{etc.},$$

quae expressio pariter ex infinitis membris constat, in quorum singulis tam numerator quam denominator etiam binis factoribus constat, prorsus uti nostra expressio pro O inventa, unde haud difficulter litterae p, q et m ita definiri poterunt, ut prodeat $O=\frac{P}{Q}$, siquidem littera n utrinque eundem significatum retinet; hocque modo valor litterae O saltem ad formulas integrales ordinarias P et Q reducetur. Hic autem probe est recordandum singulas litteras p, q, m et n numeros positivos designare debere, id quod etiam de nostris litteris a, b et c est tenendum, quandoquidem formula nostra canonica $\int\frac{x^m dx}{lx}=l(1+m)$ cum veritate consistere nequit, nisi $1+m$ fuerit numerus positivus, quia alioquin logarithmi numerorum negativorum hinc prodeuntes forent imaginarii.

12. Ad hanc conformitatem $\frac{P}{Q}$ et O constituendam sufficiet membra prima, quae sunt

$$\frac{a(a+b+c)}{(a+b)(a+c)} \quad \text{et} \quad \frac{(m+p)q}{p(m+q)},$$

ad identitatem perduxisse, propterea quod deinceps omnia sequentia membra

1) Vide *Institutionum calculi integralis* vol. I, § 360, Petropoli 1768; *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 231. Vide etiam lemma 4 Commentationis 59 supra (nota p. 50) laudatae. A. G.

sponte inter se convenient. Ista autem identitas duplici modo obtineri poterit; sumto enim $q = a$ vel statui poterit $m + q = a + b$ vel $m + q = a + c$, ita ut priori modo sit $m = b$, posteriori vero modo $m = c$; at vero tum pro priori modo erit $p = a + c$, unde sponte fiet $m + p = a + b + c$; pro posteriori vero modo, quo $m = c$, sumi debet $p = a + b$, unde denuo sponte fit $m + p = a + b + c$; quamobrem hinc geminos valores pro p et q nanciscemur, unde etiam geminae solutiones orientur, quae sunt:

$$\text{I. Solutio} \begin{cases} P = \int x^{a+c-1} dx (1-x^n)^{\frac{b-n}{n}} \\ Q = \int x^{a-1} \quad dx (1-x^n)^{\frac{b-n}{n}} \end{cases}$$

$$\text{II. Solutio} \begin{cases} P = \int x^{a+b-1} dx (1-x^n)^{\frac{c-n}{n}} \\ Q = \int x^{a-1} \quad dx (1-x^n)^{\frac{c-n}{n}} \end{cases}$$

utrinque enim erit $O = \frac{P}{Q}$, et cum sit $S = lO$, erit $S = lP - lQ$ sicque valorem ipsius S per formulas finitas expressum invenimus.

13. Circa valores autem litterarum p et q duos casus imprimis memorabiles notari convenit, quibus eos adeo absolute exhibere licet; alter enim praebet

$$\int x^{n-1} dx (1-x^n)^{\frac{m-n}{n}} = \frac{1}{m},$$

alter vero in hoc consistit, ut sit

$$\int x^{n-m-1} dx (1-x^n)^{\frac{m-n}{n}} = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

ubi π denotat 180^0 sive semiperipheriam circuli, cuius radius $= 1$. Quare cum pro nostra solutione priore sit $m = b$, videamus, utrum p et q ad istos valores absolutos reducere liceat. Hoc autem evenit, quando $b = c$ et insuper $a = n - b$, quo casu ambae solutiones inter se congruent, quem ergo casum seorsim evolvisse operae pretium erit.

EVOLUTIO CASUS QUO $c=b$ ET $a=n-b$

14. Hoc igitur casu erit formula proposita

$$S=\int\frac{x^{n-b-1}dx}{lx}\cdot\frac{(1-x^b)^2}{1-x^n};$$

tum vero vidimus esse

$$P=\int x^{n-1}dx(1-x^n)^{\frac{b-n}{n}}=\frac{1}{b}$$

et

$$Q=\int x^{n-b-1}dx(1-x^n)^{\frac{b-n}{n}}=\frac{\pi}{n\sin.\frac{b\pi}{n}},$$

quamobrem, cum sit $S=lP-lQ=l\frac{P}{Q}$, erit his valoribus substitutis

$$S=l\frac{n\sin.\frac{b\pi}{n}}{b\pi},$$

ubi evidens est esse debere $b<n$, unde sequentia exempla considerasse iuvabit.

EXEMPLUM 1 QUO $b=1$ ET $n=2$

15. Hoc ergo casu erit $\sin.\frac{b\pi}{n}=1$ hincque $S=l\frac{2}{\pi}$; quamobrem si formula proposita fuerit

$$S=\int\frac{dx}{lx}\cdot\frac{1-x}{1+x},$$

erit $S=l\frac{2}{\pi}$; at vero valorem ipsius S per logarithmos evolvendo, uti supra fecimus, ob $a=1$, $b=c=1$ et $n=2$ prodibit

$$S=\begin{cases} l1+l3+l5+l7+l9+l11+\text{etc.}\\ -2l2-2l4-2l6-2l8-2l10-\text{etc.}\\ +l3+l5+l7+l9+l11+l13+\text{etc.,}\end{cases}$$

quibus in ordinem redactis erit

$$S=l1-2l2+2l3-2l4+2l5-2l6+2l7-2l8+2l9-\text{etc.}$$

16. Vicissim igitur si proponatur ista series logarithmorum

$$s = l1 - l2 + l3 - l4 + l5 - l6 + l7 - \text{etc.},$$

eius summa assignari poterit. Cum enim sit $S - 2s = -l1 = 0$, ob $S = l\frac{2}{\pi}$ erit $s = \frac{1}{2}l\frac{2}{\pi} = l\sqrt{\frac{2}{\pi}}$; sive cum sit $\pi > 2$, erit $s = -l\sqrt{\frac{\pi}{2}}$; ista scilicet summa s erit negativa.

EXEMPLUM 2 QUO $b = 1$ ET $n = 3$

17. Hoc igitur casu, quo $a = 2$, formula integranda proposita erit

$$S = \int \frac{x\,dx}{lx}\cdot\frac{(1-x)^2}{1-x^3} = \int \frac{x\,dx}{lx}\cdot\frac{1-x}{1+x+xx};$$

deinde cum sit $\sin.\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, valor quaesitus erit $S = l\frac{3\sqrt{3}}{2\pi}$; at vero idem valor S per seriem logarithmorum expressus ob $a = 2$, $b = c = 1$ et $n = 3$ erit

$$S = \begin{cases} \quad l2 + l5 + l8 + l11 + l14 + l17 + \text{etc.} \\ -2l3 - 2l6 - 2l9 - 2l12 - 2l15 - \text{etc.} \\ +l4 + l7 + l10 + l13 + l16 + l19 + \text{etc.}; \end{cases}$$

sicque ergo erit

$$S = l2 - 2l3 + l4 + l5 - 2l6 + l7 + l8 - 2l9 + l10 + l11 - 2l12 + l13 + l14 - \text{etc.},$$

cuius ergo seriei satis regularis summa est $S = l\frac{3\sqrt{3}}{2\pi}$.

EXEMPLUM 3 QUO $b = 2$ ET $n = 3$

18. Hoc igitur casu erit $a = 1$ et formula nostra integralis fiet

$$S = \int \frac{dx}{lx}\cdot\frac{(1-xx)^2}{1-x^3} = \int \frac{dx}{lx}\cdot\frac{(1-x)(1+x)^2}{1+x+xx},$$

cuius ergo valor erit $S = l\frac{3\sqrt{3}}{4\pi}$; at vero idem valor S per seriem logarithmorum expressus ob $a = 1$, $b = c = 2$ et $n = 3$ erit

$$S = \begin{cases} \quad l1 + l4 + l7 + l10 + l13 + \text{etc.} \\ -2l3 - 2l6 - 2l9 - 2l12 - \text{etc.} \\ +l5 + l8 + l11 + l14 + \text{etc.}; \end{cases}$$

sicque ergo erit

$$S = l1 - 2l3 + l4 + l5 - 2l6 + l7 + l8 - 2l9 + l10 + l11 - \text{etc.},$$

cuius seriei summa est $S = l\frac{3\sqrt{3}}{4\pi}$; unde, cum sit $S = l\frac{P}{Q}$, erit

$$\frac{P}{Q} = \frac{1\cdot 5}{3\cdot 3}\cdot\frac{4\cdot 8}{6\cdot 6}\cdot\frac{7\cdot 11}{9\cdot 9}\cdot\frac{10\cdot 14}{12\cdot 12}\cdot\text{etc.},$$

cuius ergo valor erit $= \frac{3\sqrt{3}}{4\pi}$.

EXEMPLUM 4 QUO $b=1$ ET $n=4$

19. Hinc ergo ob $a=3$ formula nostra integralis erit

$$S = \int \frac{xx\,dx}{lx}\cdot\frac{(1-x)^2}{1-x^4} = \int \frac{xx\,dx}{lx}\cdot\frac{1-x}{(1+x)(1+xx)},$$

cuius ergo valor erit $= l\frac{2\sqrt{2}}{\pi}$; at vero idem valor per seriem logarithmorum ob $a=3$, $b=c=1$ et $n=4$ hoc modo exprimetur

$$S = \begin{cases} \quad l3 + l7 + l11 + l15 + \text{etc.} \\ -2l4 - 2l8 - 2l12 - \text{etc.} \\ +l5 + l9 + l13 + \text{etc.}; \end{cases}$$

hincque ergo erit

$$\frac{P}{Q} = \frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\text{etc.} = \frac{2\sqrt{2}}{\pi}.$$

EXEMPLUM 5 QUO $b=3$ ET $n=4$

20. Hoc ergo casu erit $a=1$ et formula nostra integralis fiet

$$S = \int \frac{dx}{lx}\cdot\frac{(1-x^3)^2}{1-x^4},$$

cuius valor erit $S = l\frac{2\sqrt{2}}{3\pi}$, qui etiam hoc modo per seriem logarithmorum exprimetur

$$S = \begin{cases} \quad l1 + l5 + l9 + l13 + l17 + \text{etc.} \\ -2l4 - 2l8 - 2l12 - 2l16 - \text{etc.} \\ +l7 + l11 + l15 + l19 + \text{etc.}; \end{cases}$$

hinc ergo fiet

$$\frac{P}{Q}=\frac{1\cdot 7}{4\cdot 4}\cdot\frac{5\cdot 11}{8\cdot 8}\cdot\frac{9\cdot 15}{12\cdot 12}\cdot\frac{13\cdot 19}{16\cdot 16}\cdot\text{etc.}=\frac{2\sqrt{2}}{3\pi}.$$

21. Praeter hos autem casus, quibus ambas formulas P et Q simul integrationem admittere observavimus, pro certo affirmare licet nullos alios insuper dari, quibus hoc eveniat. Interim tamen dantur innumerabiles alii casus, quibus valor nostrae formulae integralis S absolute sine formulis integralibus assignari potest, etiamsi neutra formularum P et Q seorsim integrari queat; qui casus cum per se sint notatu dignissimi, iis investigandis sequens problema destinemus.

PROBLEMA

22. *Investigare casus, quibus formulae integralis propositae valorem S absolute sine formulis integralibus exprimere licet.*

SOLUTIO

Totum ergo negotium huc redit, ut eiusmodi relationes inter exponentes a, b, c et n eruantur, quibus fractio supra adhibita $\frac{P}{Q}$ absolute exprimi queat, quamvis neutra harum formularum seorsim integrationem admittat; tum enim formulae propositae valor quaesitus erit $S=l\frac{P}{Q}$. Verum istam fractionem $\frac{P}{Q}$ vidimus designare istud productum in infinitum excurrens

$$\frac{P}{Q}=\frac{a(a+b+c)}{(a+b)(a+c)}\cdot\frac{(a+n)(a+b+c+n)}{(a+b+n)(a+c+n)}\cdot\frac{(a+2n)(a+b+c+2n)}{(a+b+2n)(a+c+2n)}\cdot\text{etc.}$$

23. Nunc vero meminisse iuvabit tam sinus quam cosinus angulorum per huiusmodi producta infinita exprimi solere; cum enim sit

$$\sin.\frac{p\pi}{2r}=\frac{p\pi}{2r}\cdot\frac{4rr-pp}{4rr}\cdot\frac{16rr-pp}{16rr}\cdot\frac{36rr-pp}{36rr}\cdot\text{etc.},$$

erit duabus huiusmodi expressionibus combinandis

$$\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}}=\frac{p}{q}\cdot\frac{4rr-pp}{4rr-qq}\cdot\frac{16rr-pp}{16rr-qq}\cdot\frac{36rr-pp}{36rr-qq}\cdot\frac{64rr-pp}{64rr-qq}\cdot\text{etc.}$$

Quare si superior expressio pro $\frac{P}{Q}$ inventa ad hanc formam revocari queat, tum utique erit

$$S = l\sin.\frac{p\pi}{2r} - l\sin.\frac{q\pi}{2r}.$$

Quo autem ista reductio facilius succedat, posteriorem expressionem hac forma repraesentemus

$$\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}} = \frac{p(2r-p)}{q(2r-q)}\cdot\frac{(2r+p)(4r-p)}{(2r+q)(4r-q)}\cdot\frac{(4r+p)(6r-p)}{(4r+q)(6r-q)}\cdot\text{etc.},$$

cuius expressionis membra manifesto ita progrediuntur, ut singuli factores tam numeratorum quam denominatorum continuo eodem incremento $2r$ augeantur. Quare cum in expressione $\frac{P}{Q}$ singuli factores capiant incrementum n, statui debebit $n = 2r$, quo notato sufficiet prima membra ad conformitatem redigere, id quod eveniet sumendo

$$a = p,\ a+b+c = 2r-p,\ a+b = q,\ a+c = 2r-q,$$

unde singulae litterae colliguntur

$$1^0.\ a = p, \qquad 2^0.\ b = q-p, \qquad 3^0.\ c = 2r-p-q$$

existente $n = 2r$. Hinc autem operae pretium erit notasse fore

$$2a+b+c = 2r = n,$$

ita ut formula nostra generalis ad casum hunc semper accommodari queat, si modo fuerit $n = 2a+b+c$; tum enim fit $p = a$, $q = a+b$ et $2r = 2a+b+c$.

24. Quodsi vero formula nostra generalis evolvatur ac loco n scribatur iste valor $2a+b+c$, ea induet hanc formam

$$S = \int\frac{dx}{xlx}\cdot\frac{x^a - x^{a+b} - x^{a+c} + x^{a+b+c}}{1-x^{2a+b+c}},$$

cuius ergo valor, si loco p, q et r modo inventi valores scribantur, erit

$$S = l\frac{P}{Q} = l\sin.\frac{a\pi}{2a+b+c} - l\sin.\frac{(a+b)\pi}{2a+b+c},$$

quae formula utique ita est absoluta, ut nullam amplius formulam integralem involvat, prorsus uti desideratur. Patet igitur casum ante tractatum in hoc casu non contineri; cum enim in illo fuisset $a=n-b$ et $c=b$, hinc fiet $2a+b+c=2n$, cum praesenti casu sit $2a+b+c=n$.

25. Quodsi iam in hac expressione litteras p, q et r in calculum introducamus, formula nostra integralis ad hanc speciem reducetur

$$S=\int\frac{dx}{xlx}\cdot\frac{x^p-x^q-x^{2r-q}+x^{2r-p}}{1-x^{2r}},$$

cuius igitur valor ab $x=0$ ad $x=1$ extensus erit

$$S=l\sin.\frac{p\pi}{2r}-l\sin.\frac{q\pi}{2r},$$

ubi manifestum est hanc expressionem eandem manere, etiamsi loco p scribatur $2r-p$, loco q vero $2r-q$, propterea quod

$$\sin.\frac{(2r-p)\pi}{2r}=\sin.\frac{p\pi}{2r}\quad\text{et}\quad\sin.\frac{(2r-q)\pi}{2r}=\sin.\frac{q\pi}{2r};$$

at vero ipsa formula integralis facta sive alterutra substitutione sive utraque coniunctim prorsus non variatur.

26. Quodsi loco p et q scribamus $r-p$ et $r-q$, illi sinus transmutantur in cosinus; tum autem ipsa formula integralis erit

$$S=\int\frac{dx}{xlx}\cdot\frac{x^{r-p}-x^{r-q}-x^{r+q}+x^{r+p}}{1-x^{2r}},$$

cuius valor nunc erit

$$=l\cos.\frac{p\pi}{2r}-l\cos.\frac{q\pi}{2r},$$

ubi iterum manifestum est nullam mutationem oriri, sive litterae p et q valores habeant positivos sive negativos.

COROLLARIUM 1

27. Cum igitur his casibus neutra formularum integralium P et Q integrationem actu admittat, eo magis notatu dignum hic occurrit, quod nihilo-

minus valor fractionis $\frac{P}{Q}$ absolute exprimi possit, cum per sinus sit

$$\frac{P}{Q}=\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}}.$$

Cum igitur hoc casu sit $a=p$, $b=q-p$, $c=2r-p-q$ et $n=2r$, valores integrales pro P et Q supra (§ 12) exhibiti in sequentes abibunt formas

$$P=\int\frac{x^{2r-q-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}.$$

Quicunque ergo valores exponentibus tribuantur, semper erit $\frac{P}{Q}=\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}}$.

COROLLARIUM 2

28. Quoniam hic loco p et q scribere licet $2r-p$ et $2r-q$, hinc quaternas formulas integrales exhibere possumus, ita ut pro singulis sit

$$\frac{P}{Q}=\frac{\sin.\frac{p\pi}{2r}}{\sin.\frac{q\pi}{2r}},$$

qui quaterni valores ita se habebunt

$$\text{I.}\quad P=\int\frac{x^{2r-q-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}},$$

$$\text{II.}\quad P=\int\frac{x^{2r-q-1}dx}{(1-x^{2r})^{2-\frac{p+q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}dx}{(1-x^{2r})^{2-\frac{p+q}{2r}}},$$

$$\text{III.}\quad P=\int\frac{x^{q-1}dx}{(1-x^{2r})^{\frac{p+q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{\frac{p+q}{2r}}},$$

$$\text{IV.}\quad P=\int\frac{x^{q-1}dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}}.$$

COROLLARIUM 3

29. Quodsi hic loco p et q scribamus $r-p$ et $r-q$, quo pacto sinus in cosinus transmutantur, quaternas impetrabimus formulas integrales pro P et Q ita comparatas, ut pro omnibus sit

$$\frac{P}{Q}=\frac{\cos.\frac{p\pi}{2r}}{\cos.\frac{q\pi}{2r}},$$

qui quaterni valores erunt

$$\text{I.}\quad P=\int\frac{x^{r+q-1}dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{r-p-1}dx}{(1-x^{2r})^{1+\frac{q-p}{2r}}},$$

$$\text{II.}\quad P=\int\frac{x^{r+q-1}dx}{(1-x^{2r})^{1+\frac{p+q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{r+p-1}dx}{(1-x^{2r})^{1+\frac{p+q}{2r}}},$$

$$\text{III.}\quad P=\int\frac{x^{r-q-1}dx}{(1-x^{2r})^{1-\frac{p+q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{r-p-1}dx}{(1-x^{2r})^{1-\frac{p+q}{2r}}},$$

$$\text{IV.}\quad P=\int\frac{x^{r-q-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{r+p-1}dx}{(1-x^{2r})^{1+\frac{p-q}{2r}}},$$

quae quaternae formulae tam pulchre inter se conspirant, ut aliter non discrepent nisi ratione signorum, quibus litterae p et q sunt affectae.

COROLLARIUM 4

30. Hae autem formulae prorsus sunt diversae ab illis, quas supra in evolutione § 14 habuimus, ubi erat $\frac{P}{Q}=\frac{n}{b\pi}\sin.\frac{b\pi}{n}$; quod discrimen quo clarius ob oculos ponatur, loco b et n scribamus p et $2r$, ut fiat $\frac{P}{Q}=\frac{2r}{p\pi}\sin.\frac{p\pi}{2r}$; tum autem fit

$$P=\int\frac{x^{2r-1}\,dx}{(1-x^{2r})^{1-\frac{p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}\,dx}{(1-x^{2r})^{1-\frac{p}{2r}}},$$

quae formulae actu integrationem admittent, dum colligitur

$$P=\frac{1}{p}\quad\text{et}\quad Q=\frac{\pi}{2r\sin.\frac{p\pi}{2r}}.$$

COROLLARIUM 5

31. Quodsi in formulis penultimi corollarii capiamus $q=0$, ut fiat $\frac{P}{Q}=\cos.\frac{p\pi}{2r}$, binas tantum pro hoc casu diversas formulas pro P et Q nanciscemur, quae sunt

$$\text{I.}\quad P=\int\frac{x^{r-1}dx}{(1-x^{2r})^{1-\frac{p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{r-p-1}dx}{(1-x^{2r})^{1-\frac{p}{2r}}},$$

$$\text{II.}\quad P=\int\frac{x^{r-1}dx}{(1-x^{2r})^{1+\frac{p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{r+p-1}dx}{(1-x^{2r})^{1+\frac{p}{2r}}}.$$

Sin autem in formulis antepenultimi corollarii statuamus $q=r$, ut prodeat $\frac{P}{Q}=\sin.\frac{p\pi}{2r}$, iterum prodibunt binae formulae pro P et Q, quae sunt

$$\text{I.}\quad P=\int\frac{x^{r-1}dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{2r}}},$$

$$\text{II.}\quad P=\int\frac{x^{r-1}dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{2r}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{2r}}}.$$

COROLLARIUM 6

32. Quodsi in formulis Corollarii 2 statuamus $q=r-p$, ut fiat $\sin.\frac{q\pi}{2r}=\cos.\frac{p\pi}{2r}$, habebitur $\frac{P}{Q}=\text{tang.}\frac{p\pi}{2r}$ et quaterni valores pro formulis P et Q erunt

$$\text{I.}\quad P=\int\frac{x^{r+p-1}dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{r}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{\frac{1}{2}+\frac{p}{r}}},$$

$$\text{II.}\quad P=\int\frac{x^{r+p-1}dx}{(1-x^{2r})^{\frac{3}{2}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}dx}{(1-x^{2r})^{\frac{3}{2}}},$$

$$\text{III.}\quad P=\int\frac{x^{r-p-1}dx}{(1-x^{2r})^{\frac{1}{2}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}dx}{(1-x^{2r})^{\frac{1}{2}}},$$

$$\text{IV.}\quad P=\int\frac{x^{r-p-1}\,dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{r}}}\quad\text{et}\quad Q=\int\frac{x^{2r-p-1}dx}{(1-x^{2r})^{\frac{3}{2}-\frac{p}{r}}}.$$

COROLLARIUM 7

33. Plurimum autem etiam intererit nosse ipsam formulam integralem S pro his casibus, quibus fit simpliciter vel $\frac{P}{Q}=\cos.\frac{p\pi}{2r}$ vel $\frac{P}{Q}=\sin.\frac{p\pi}{2r}$ vel $\frac{P}{Q}=\operatorname{tang}.\frac{p\pi}{2r}$, fieri pro primo

$$S=\int\frac{dx}{xlx}\cdot\frac{x^{r-p}-2x^r+x^{r+p}}{1-x^{2r}}=l\cos.\frac{p\pi}{2r},$$

pro secundo casu

$$S=\int\frac{dx}{xlx}\cdot\frac{x^p-2x^r+x^{2r-p}}{1-x^{2r}}=l\sin.\frac{p\pi}{2r},$$

pro tertio autem casu

$$S=\int\frac{dx}{xlx}\cdot\frac{x^p-x^{r-p}-x^{r+p}+x^{2r-p}}{1-x^{2r}}=l\operatorname{tang}.\frac{p\pi}{2r},$$

quae postrema formula reducitur ad hanc

$$\int\frac{dx}{xlx}\cdot\frac{x^p-x^{r-p}}{1+x^r}=l\operatorname{tang}.\frac{p\pi}{2r},$$

quae est eadem integratio, quam non ita pridem ex diversissimis principiis elicueram.[1])

SCHOLION

34. Postremo autem circa omnes has varias formulas integrales probe notetur eas, in quibus exponens denominatoris reperitur unitate maior, utpote incongruas reiiciendas esse, propterea quod earum valores integrati posito $x=1$ evadant infiniti, quod quidem, cum in utraque formula P et Q simul eveniat, non impedit, quominus fractio $\frac{P}{Q}$ assignatum obtineat valorem; sed quia eum hinc definire non licet, etiam istiusmodi formulae optatum usum non praestant. Commode autem evenit, ut plures formulae adsint, ex quibus valorem verum derivare liceat.

1) Vide § 42 Commentationis 463 supra (nota 1 p. 53) laudatae. A. G.

THEOREMES ANALYTIQUES

EXTRAITS DE DIFFERENTES LETTRES DE M. EULER A M. LE MARQUIS DE CONDORCET

Commentatio 521 indicis ENESTROEMIANI
Mémoires de l'académie des sciences de Paris 1778 (1781), p. 603—614
Résumé ibidem p. 42

RESUME

Ces deux théorèmes ont été proposés et démontrés par M. EULER; l'un donne pour une valeur déterminée l'expression de l'intégrale de plusieurs fonctions dont on ne peut connoître l'intégrale pour une valeur quelconque. Quoiqu'il ne soit ici question que de deux fonctions, il est aisé de voir que ce ne sont que des exemples d'une méthode plus générale, qui embrasse une classe de fonctions très-étendue. Le second théorème donne en un seul produit de facteurs, l'expression de la somme des carrés des coëfficiens de la formule du binome élevé à une puissance quelconque. La démonstration de M. EULER est beaucoup plus générale que l'énoncé du théorème, puisqu'elle donne une expression semblable pour la somme des coëfficiens d'une puissance du binome multipliée par les coëfficiens successifs d'une autre puissance quelconque.

On a joint à ce mémoire une autre démonstration des mêmes théorèmes, trouvée avant de connoître celle de M. EULER; l'auteur de cette démonstration a espéré qu'on ne le soupçonneroit pas de la présomption d'avoir voulu comparer son travail à celui d'un grand homme, dont il s'honore d'être l'admirateur et le disciple.

LETTRE DU $\frac{3}{14}$ NOVEMBRE 1775

L'intégrale de cette formule,

$$\frac{x^m - x^n}{lx} \cdot \frac{\partial x}{x},$$

prise depuis $x = 0$ *jusqu'à* $x = 1$, *est* $= l\frac{m}{n}$.

L'intégrale de cette formule,

$$\frac{x^{m-1}\partial x}{(1+x^n)lx},$$

prise depuis $x=0$ *jusqu'à* $x=\infty$, *est* $=l.\text{tang.}\frac{m\pi}{n}$, *où* π *marque l'angle de* 180 *degrés.*

LETTRE DU 2 FEVRIER 1776

DEMONSTRATION DES DEUX THEOREMES PRECEDENS

Soit Q une fonction quelconque des deux variables x et y, et qu'on cherche la quantité Z, telle que

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right)=Q,$$

où il s'agit d'une double intégration; l'une, où la seule x est prise pour variable, et l'autre, où la seule y varie; la première devra être étendue depuis $x=0$ jusqu'à $x=1$ et l'autre depuis $y=0$ jusqu'à $y=n$; par la nature de telles formules, on aura donc d'une double manière ou

$$Z=\int\partial x\int Q\partial y,$$

ou

$$Z=\int\partial y\int Q\partial x,$$

Maintenant, qu'on suppose

$$Q=x^y,$$

et on aura

$$\int Q\partial y=\frac{x^y}{lx}-\frac{1}{lx},$$

afin que cette intégrale évanouisse lorsque $y=0$. Soit donc à présent $y=n$, et nous aurons

$$\int Q\partial y=\frac{x^n-1}{lx}$$

et partant

$$Z=\int\frac{(x^n-1)\partial x}{lx};$$

ensuite, nous aurons

$$\int Q\partial x=\frac{x^{y+1}}{y+1},$$

qui évanouit lorsque $x=0$; posant donc $x=1$, il en résulte

$$\int Q\partial x=\frac{1}{y+1},$$

et de là,

$$Z=\int\frac{\partial y}{y+1}=l(y+1),$$

(expression qui disparoît lorsque $y=0$). Qu'on fasse donc $y=n$, et l'on aura $Z=l(n+1)$; par conséquent, il est certain que cette intégrale $\int\frac{\partial x(x^n-1)}{lx}$, prise depuis $x=0$ jusqu'à $x=1$, est $l(n+1)$.

Pour l'autre formule intégrale plus compliquée que je vous avois communiquée, j'avois supposé

$$Q=\frac{x^{m-y}+x^{m+y}}{(1+x^{2m})x};$$

de là, prenant d'abord x constante à cause de

$$\int x^{m-y}\partial y=-\frac{x^{m-y}}{lx} \quad \text{et de} \quad \int x^{m+y}\partial y=\frac{x^{m+y}}{lx},$$

on aura

$$\int Q\partial y=\frac{x^{m+y}-x^{m-y}}{(1+x^{2m})xlx},$$

ce qui devient $=0$ posant $y=0$. Faisant donc $y=n$, on aura

$$\int Q\partial y=\frac{x^{m+n}-x^{m-n}}{(1+x^{2m})xlx}$$

et partant

$$Z=\int\frac{(x^{m+n}-x^{m-n})\partial x}{(1+x^{2m})xlx}.$$

L'autre intégration donne d'abord

$$\int Q\partial x=\int\frac{(x^{m-y}+x^{m+y})\partial x}{(1+x^{2m})x},$$

dont l'intégrale doit être étendue depuis $x=0$ jusqu'à $x=1$; or, pour ce

cas, j'ai démontré autrefois[1]) que cette intégrale se réduit à cette forme,

$$\frac{\pi}{2m\cos.\frac{\pi y}{2m}},$$

d'où nous tirons

$$Z=\int\frac{\pi\partial y}{2m\cos.\frac{\pi y}{2m}}.$$

Pour cette forme, posons $\frac{\pi y}{2m}=\varphi$, pour avoir

$$Z=\int\frac{\partial\varphi}{\cos.\varphi}=\int\frac{\partial\varphi}{\sin.(90^d+\varphi)},$$

dont l'intégrale est $l.\operatorname{tang.}\left(45^d+\frac{1}{2}\varphi\right)$ et partant, $Z=l.\operatorname{tang.}\left(45^d+\frac{\pi y}{4m}\right)$ qui en effet évanouit prenant $y=0$. Faisons donc $y=n$, et nous aurons

$$Z=l.\operatorname{tang.}\left(45^d+\frac{\pi n}{4m}\right);$$

d'où il est clair que sous les conditions présentes, on aura

$$\int\frac{(x^{m+n-1}-x^{m-n-1})\partial x}{(1+x^{2m})lx}\begin{cases}\text{depuis } x=0\\ \text{jusqu'à } x=1\end{cases}=l.\operatorname{tang.}\left(45^d+\frac{\pi n}{4m}\right).$$

Par ces deux exemples, on verra aisément que cette spéculation mérite toute l'attention des géomètres. La première idée qui m'a conduit à cette recherche, étoit tirée d'un principe entièrement différent, que voici. J'avois considéré cette formule

$$\int\frac{(x-1)\partial x}{lx},$$

où, au lieu de lx, j'ai écrit cette valeur $\frac{x^\omega-1}{\omega}$, en supposant ω infiniment petit, ou bien

$$lx=i(x^{\frac{1}{i}}-1),$$

1) Voir § 3 du mémoire 463 (suivant *l'Index* d'ENESTRÖM): *De valore formulae integralis* $\int\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}}\cdot\frac{dz}{z}(lz)^\mu$ *casu quo post integrationem ponitur* $z=1$, Novi comment. acad. sc. Petrop. **19** (1774), 1775, p. 30; *LEONHARDI EULERI Opera omnia*, series I, vol. 17. A. G.

en prenant pour i un nombre infiniment grand. Qu'on pose à présent $x^{\frac{1}{i}} = z$, ou bien $x = z^i$, où il faut remarquer que les termes de l'intégration $x = 0$ et $x = 1$ se réduisent à $z = 0$ et à $z = 1$; cette valeur étant substituée, transforme notre formule en celle-ci, $\frac{(z^i - 1)z^{i-1}\partial z}{z-1}$; or, la fraction $\frac{z^i - 1}{z-1}$ ou bien $\frac{1-z^i}{1-z}$, se réduit à la série

$$1 + z + z^2 + z^3 + \cdots + z^{i-1},$$

qui étant multipliée [par $z^{i-1}\partial z$] et intégrée, donne

$$\frac{z^i}{i} + \frac{z^{i+1}}{i+1} + \frac{z^{i+2}}{i+2} + \frac{z^{i+3}}{i+3} + \cdots + \frac{z^{2i-1}}{2i-1},$$

et, posant $z = 1$, la valeur cherchée sera

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \frac{1}{i+4} + \cdots + \frac{1}{2i-1},$$

dont la valeur est $l2$, de sorte que

$$\int \frac{(x-1)\partial x}{lx} \begin{Bmatrix} \text{depuis } x = 0 \\ \text{jusqu'à } x = 1 \end{Bmatrix} \text{ est } = l2.$$

Pour démontrer la somme de la série trouvée, qu'on appellera A, on n'a qu'à remarquer que

$$\begin{aligned} A = 1 &+ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{i-1} \\ &+ \frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \cdots + \frac{1}{2i-1} \\ &- \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{1-i}\right), \end{aligned}$$

où, parce que la série supérieure contient deux fois plus de termes que l'inférieure, on n'a qu'à soustraire chaque terme de la dernière de la supérieure alternativement, et l'on aura

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots$$
$$+ \frac{1}{i-1} + \frac{1}{i} + \frac{1}{i+1} + \cdots + \frac{1}{2i-1}$$
$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \text{etc.}$$

ou bien

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \text{etc.} = l2.$$

AUTRE THEOREME

En prenant les lettres α, β, γ, δ, etc. pour marquer les coëfficiens d'un binome élevé à l'exposant n, de sorte que

$$(1+x)^n = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.},$$

on aura toujours

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.} = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdots \frac{4n-2}{n}.$$

Par exemple, si $n = 6$, on aura $\alpha = 6$, $\beta = 15$, $\gamma = 20$, $\delta = 15$, $\varepsilon = 6$, $\zeta = 1$ et les suivans $= 0$; et partant, on aura

$$1 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5} \cdot \frac{22}{6},$$

dont la démonstration directe me paroît extrêmement difficile.

LETTRE DU $\frac{12}{23}$ SEPTEMBRE 1776

DEMONSTRATION DE CE THEOREME

. . . en supposant

$$(1+z)^n = 1 + \left(\frac{n}{1}\right) z + \left(\frac{n}{2}\right) z^2 + \left(\frac{n}{3}\right) z^3 + \text{etc.};$$

d'où l'on voit que $\left(\frac{n}{0}\right) = 1$, aussi bien que $\left(\frac{n}{n}\right)$, et de là, il s'ensuit que $\left(\frac{n}{p}\right) = \left(\frac{n}{n-p}\right)$; outre cela, il est clair que la valeur de la formule $\left(\frac{n}{p}\right)$ est

toujours égale à zéro, tant dans les cas où p est un nombre négatif, que dans ceux où il est un nombre plus grand que n, ce qui s'entend des nombres entiers; ensuite, on sait que la valeur développée de ce caractère $\left(\frac{n}{p}\right)$ est

$$=\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdots\frac{n-p+1}{p}.$$

Cela posé, si nous passons aux coëfficiens de la puissance suivante $(1+z)^{n+1}$, on sait qu'on aura

$$\left(\frac{n+1}{p+1}\right)=\left(\frac{n}{p}\right)+\left(\frac{n}{p+1}\right);$$

de sorte que réciproquement

$$\left(\frac{n}{p+1}\right)+\left(\frac{n}{p+2}\right)=\left(\frac{n+1}{p+2}\right);$$

ajoutons ces deux équations ensemble, et nous aurons

$$\left(\frac{n}{p}\right)+2\left(\frac{n}{p+1}\right)+\left(\frac{n}{p+2}\right)=\left(\frac{n+1}{p+1}\right)+\left(\frac{n+1}{p+2}\right)=\left(\frac{n+2}{p+2}\right);$$

de la même manière, nous aurons

$$\left(\frac{n}{p+1}\right)+2\left(\frac{n}{p+2}\right)+\left(\frac{n}{p+3}\right)=\left(\frac{n+2}{p+3}\right);$$

cette équation, ajoutée à la précédente, donne

$$\left(\frac{n}{p}\right)+3\left(\frac{n}{p+1}\right)+3\left(\frac{n}{p+2}\right)+\left(\frac{n}{p+3}\right)=\left(\frac{n+2}{p+2}\right)+\left(\frac{n+2}{p+3}\right)=\left(\frac{n+3}{p+3}\right);$$

ensuite

$$\left(\frac{n}{p+1}\right)+3\left(\frac{n}{p+2}\right)+3\left(\frac{n}{p+3}\right)+\left(\frac{n}{p+4}\right)=\left(\frac{n+3}{p+4}\right),$$

qui, encore ajoutée à la précédente, donne

$$\left(\frac{n}{p}\right)+4\left(\frac{n}{p+1}\right)+6\left(\frac{n}{p+2}\right)+4\left(\frac{n}{p+3}\right)+\left(\frac{n}{p+4}\right)=\left(\frac{n+3}{p+3}\right)+\left(\frac{n+3}{p+4}\right)=\left(\frac{n+4}{p+4}\right),$$

et de là, il est aisé à conclure qu'on aura en général

$$1\left(\frac{n}{p}\right)+\left(\frac{m}{1}\right)\left(\frac{n}{p+1}\right)+\left(\frac{m}{2}\right)\left(\frac{n}{p+2}\right)+\left(\frac{m}{3}\right)\left(\frac{n}{p+3}\right)+\text{etc.}=\left(\frac{n+m}{p+m}\right).$$

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots$$
$$+ \frac{1}{i-1} + \frac{1}{i} + \frac{1}{i+1} + \cdots + \frac{1}{2i-1}$$
$$- 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \text{etc.}$$

ou bien

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \text{etc.} = l2.$$

AUTRE THEOREME

En prenant les lettres α, β, γ, δ, etc. pour marquer les coëfficiens d'un binome élevé à l'exposant n, de sorte que

$$(1+x)^n = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.},$$

on aura toujours

$$1 + \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.} = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdots \frac{4n-2}{n}.$$

Par exemple, si $n = 6$, on aura $\alpha = 6$, $\beta = 15$, $\gamma = 20$, $\delta = 15$, $\varepsilon = 6$, $\zeta = 1$ et les suivans $= 0$; et partant, on aura

$$1 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1 = \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5} \cdot \frac{22}{6},$$

dont la démonstration directe me paroît extrêmement difficile.

LETTRE DU $\frac{12}{23}$ SEPTEMBRE 1776

DEMONSTRATION DE CE THEOREME

. . . en supposant

$$(1+z)^n = 1 + \left(\frac{n}{1}\right)z + \left(\frac{n}{2}\right)z^2 + \left(\frac{n}{3}\right)z^3 + \text{etc.};$$

d'où l'on voit que $\left(\frac{n}{0}\right) = 1$, aussi bien que $\left(\frac{n}{n}\right)$, et de là, il s'ensuit que $\left(\frac{n}{p}\right) = \left(\frac{n}{n-p}\right)$; outre cela, il est clair que la valeur de la formule $\left(\frac{n}{p}\right)$ est

toujours égale à zéro, tant dans les cas où p est un nombre négatif, que dans ceux où il est un nombre plus grand que n, ce qui s'entend des nombres entiers; ensuite, on sait que la valeur développée de ce caractère $\left(\frac{n}{p}\right)$ est

$$=\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdots\frac{n-p+1}{p}.$$

Cela posé, si nous passons aux coëfficiens de la puissance suivante $(1+z)^{n+1}$, on sait qu'on aura

$$\left(\frac{n+1}{p+1}\right)=\left(\frac{n}{p}\right)+\left(\frac{n}{p+1}\right);$$

de sorte que réciproquement

$$\left(\frac{n}{p+1}\right)+\left(\frac{n}{p+2}\right)=\left(\frac{n+1}{p+2}\right);$$

ajoutons ces deux équations ensemble, et nous aurons

$$\left(\frac{n}{p}\right)+2\left(\frac{n}{p+1}\right)+\left(\frac{n}{p+2}\right)=\left(\frac{n+1}{p+1}\right)+\left(\frac{n+1}{p+2}\right)=\left(\frac{n+2}{p+2}\right);$$

de la même manière, nous aurons

$$\left(\frac{n}{p+1}\right)+2\left(\frac{n}{p+2}\right)+\left(\frac{n}{p+3}\right)=\left(\frac{n+2}{p+3}\right);$$

cette équation, ajoutée à la précédente, donne

$$\left(\frac{n}{p}\right)+3\left(\frac{n}{p+1}\right)+3\left(\frac{n}{p+2}\right)+\left(\frac{n}{p+3}\right)=\left(\frac{n+2}{p+2}\right)+\left(\frac{n+2}{p+3}\right)=\left(\frac{n+3}{p+3}\right);$$

ensuite

$$\left(\frac{n}{p+1}\right)+3\left(\frac{n}{p+2}\right)+3\left(\frac{n}{p+3}\right)+\left(\frac{n}{p+4}\right)=\left(\frac{n+3}{p+4}\right),$$

qui, encore ajoutée à la précédente, donne

$$\left(\frac{n}{p}\right)+4\left(\frac{n}{p+1}\right)+6\left(\frac{n}{p+2}\right)+4\left(\frac{n}{p+3}\right)+\left(\frac{n}{p+4}\right)=\left(\frac{n+3}{p+3}\right)+\left(\frac{n+3}{p+4}\right)=\left(\frac{n+4}{p+4}\right),$$

et de là, il est aisé à conclure qu'on aura en général

$$1\left(\frac{n}{p}\right)+\left(\frac{m}{1}\right)\left(\frac{n}{p+1}\right)+\left(\frac{m}{2}\right)\left(\frac{n}{p+2}\right)+\left(\frac{m}{3}\right)\left(\frac{n}{p+3}\right)+\text{etc.}=\left(\frac{n+m}{p+m}\right).$$

Voilà donc une progression bien générale, dont chaque terme est le produit de deux coëfficiens de puissances différentes du binome, dont le terme général peut être exprimé par la formule $\left(\frac{m}{x}\right)\left(\frac{n}{p+x}\right)$, où, mettant pour x successivement les nombres 0, 1, 2, 3, 4, etc., jusqu'à ce qu'on parvienne à des termes évanouissans, la somme de toute cette progression sera infailliblement $=\left(\frac{n+m}{p+m}\right)=\left(\frac{n+m}{n-p}\right)$. C'est de là que résulte le théorème que je vous ai communiqué, en faisant $m=n$ et $p=0$, de sorte qu'il est un cas infiniment plus particulier que la série que je viens de sommer ici. Dans ce cas, on aura cette sommation:

$$1^2+\left(\frac{n}{1}\right)^2+\left(\frac{n}{2}\right)^2+\left(\frac{n}{3}\right)^2+\text{etc.}=\left(\frac{2n}{n}\right);$$

or, cette formule développée donne

$$\frac{2n}{1}\cdot\frac{2n-1}{2}\cdot\frac{2n-2}{3}\cdot\frac{2n-3}{4}\cdots\frac{n+1}{n},$$

ce qui, comme il est aisé à démontrer, est égal à

$$\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4n-2}{n}.$$

Il est fort remarquable que cette sommation a aussi lieu, lors même que les exposans m et n sont des fractions quelconques, pourvu que, par la voie d'interpolation, on puisse assigner la juste valeur de $\left(\frac{m+n}{m+p}\right)$; et si le développement n'a pas lieu dans ce cas, il faut recourir à des formules intégrales; or, posant pour abréger $l\frac{1}{x}=u$, on aura toujours

$$\left(\frac{m+n}{m+p}\right)=\frac{\int u^{m+n}\partial x}{\int u^{m+p}\partial x\cdot\int u^{n-p}\partial x}\left\{\begin{matrix}\text{de } x=0\\ \text{à } x=1\end{matrix}\right\};$$

or, si λ marque un nombre entier positif quelconque, on sait qu'il y aura

$$\int u^\lambda\partial x=1\cdot2\cdot3\cdot4\cdots\lambda,$$

et de là, on tirera

$$\int u^{\lambda+1}\partial x=(\lambda+1)\int u^\lambda\partial x,$$

$$\int u^{\lambda+2}\partial x=(\lambda+1)(\lambda+2)\int u^\lambda\partial x,$$

etc.

et cette réduction aura toujours lieu, quelque nombre qu'on prenne pour λ. Prenant donc $\lambda = -\frac{1}{2}$, j'ai démontré autrefois[1]) qu'on aura

$$\int \frac{\partial x}{\sqrt{u}} = \sqrt{\pi} \quad \text{et} \quad \int \partial x \sqrt{u} = \frac{1}{2}\sqrt{\pi},$$

π désignant la circonférence d'un cercle dont le diamètre $= 1$. Maintenant, si l'on met $m = n$ et $p = 0$, puisque les coëfficiens de $(1+z)^{\frac{1}{2}}$ sont

$$1, \quad +\frac{1}{2}, \quad -\frac{1\cdot 1}{2\cdot 4}, \quad +\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, \quad -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}, \quad \text{etc.},$$

nous en tirons cette série des carrés,

$$1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 1}{2\cdot 4}\right)^2 + \left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^2 + \text{etc.}$$

dont la somme sera

$$\frac{\int u\,\partial x}{\int \partial x \sqrt{u} \cdot \int \partial x \sqrt{u}} = \frac{4}{\pi},$$

à cause de

$$\int u\,\partial x = 1 \quad \text{et} \quad \int \partial x \sqrt{u} = \frac{1}{2}\sqrt{\pi},$$

ce qui s'accorde parfaitement avec la somme qu'on trouve par la voie de l'approximation.

J'ai cru pouvoir joindre ici une autre démonstration de deux des théorèmes précédens, quoique la méthode qui y est employée, soit fort inférieure à celle de M. EULER; mais il peut être quelquefois utile de voir comment différentes routes peuvent conduire aux mêmes vérités. D'ailleurs, M. EULER ayant daigné honorer ces recherches de son approbation, c'est lui donner une marque de mon respect que de les rendre publiques.

1) Voir § 16 et 28 du mémoire 421 (suivant *l'Index* d'ENESTRÖM): *Evolutio formulae integralis $\int x^{f-1} dx (lx)^{\frac{m}{n}}$ integratione a valore $x=0$ ad $x=1$ extensa,* Novi comment. acad. sc. Petrop. **16** (1771), 1772, p. 91; *LEONHARDI EULERI Opera omnia,* series I, vol. 17, p. 316. Voir aussi § 11 du mémoire 19 (suivant *l'Index* d'ENESTRÖM): *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt,* Comment. acad. sc. Petrop. **5** (1730/1), 1738, p. 36; *LEONHARDI EULERI Opera omnia,* series I, vol. 14. A. G.

Soit la fonction $\int \frac{x^m}{lx} \cdot \frac{\partial x}{x}$ et qu'on l'intègre en série par la méthode des intégrations par parties, on aura

$$\int \frac{x^m}{lx} \cdot \frac{\partial x}{x} = x^m \left(\frac{1}{mlx} + \frac{1}{m^2 lx^2} + \frac{2}{m^3 lx^3} + \frac{2 \cdot 3}{m^4 lx^4} + \cdots \right).$$

Ainsi, la valeur de cette intégrale, prise depuis $x = B$ jusqu'à $x = A$, sera

$$(S) \qquad \left\{ \begin{array}{l} B^m \left(\frac{1}{mlB} + \frac{1}{m^2 lB^2} + \frac{2}{m^3 lB^3} + \frac{2 \cdot 3}{m^4 lB^4} + \cdots \right) \\ - A^m \left(\frac{1}{mlA} + \frac{1}{m^2 lA^2} + \frac{2}{m^3 lA^3} + \frac{2 \cdot 3}{m^4 lA^4} + \cdots \right). \end{array} \right.$$

Pour avoir maintenant la valeur de cette fonction en m, je la différencie par rapport à m et j'ai pour sa valeur:

$$\begin{aligned} B^m \Big(\frac{\partial m}{m} &+ \frac{\partial m}{m^2 lB} + \frac{2 \partial m}{m^3 lB^2} + \frac{2 \cdot 3 \partial m}{m^4 lB^3} \cdots \\ &- \frac{\partial m}{m^2 lB} - \frac{2 \partial m}{m^3 lB^2} - \frac{2 \cdot 3 \partial m}{m^4 lB^3} \cdots \Big) \\ - A^m \Big(\frac{\partial m}{m} &+ \frac{\partial m}{m^2 lA} + \frac{2 \partial m}{m^3 lA^2} + \frac{2 \cdot 3 \partial m}{m^4 lA^3} \cdots \\ &- \frac{\partial m}{m^2 lA} - \frac{2 \partial m}{m^3 lA^2} - \frac{2 \cdot 3 \partial m}{m^4 lA^3} \cdots \Big), \end{aligned}$$

valeur qui se réduit à

$$(B^m - A^m) \frac{\partial m}{m}.$$

La valeur de la série (S) sera donc

$$\int (B^m - A^m) \frac{\partial m}{m} + C;$$

et, si on suppose $A = 0$ et $B = 1$, [la valeur se réduit] à

$$\int \frac{\partial m}{m} + C = lm + C$$

C étant une constante indépendante de m; par la même raison, on aura pour valeur de $\int \frac{x^n}{lx} \cdot \frac{\partial x}{x}$, prise depuis $x = 0$ jusqu'à $x = 1$, la fonction $ln + C$; donc, la valeur $\int \frac{x^n - x^m}{lx} \cdot \frac{\partial x}{x}$, prise depuis $x = 0$ jusqu'à $x = 1$, sera $ln - lm$ ou $l \frac{n}{m}$.

On seroit parvenu à la même conclusion, sans employer les séries; en effet, le problème se réduit ici à trouver

$$\int\left(\frac{B^m}{lB}\cdot\frac{\partial B}{B}-\frac{A^m}{lA}\cdot\frac{\partial A}{A}\right);$$

or, différenciant cette fonction par rapport à m, elle devient

$$\int(B^{m-1}\partial B - A^{m-1}\partial A)\,\partial m = (B^m - A^m)\frac{\partial m}{m},$$

comme on l'a trouvé ci-dessus.

On auroit aussi trouvé immédiatement, en cherchant la valeur de

$$\int\left(\frac{B^n-B^m}{lB}\cdot\frac{\partial B}{B}-\frac{A^n-A^m}{lA}\cdot\frac{\partial A}{A}\right),$$

que cette fonction différenciée par rapport à n et à m, devient

$$(B^n - A^n)\frac{\partial n}{n} - (B^m - A^m)\frac{\partial m}{m},$$

dont l'intégrale est, lorsque $B=1$ et $A=0$, $l\frac{n}{m}+C$; mais, pour le cas de $m=n$, il est clair que cette intégrale doit être zéro; donc $C=0$; donc l'intégrale cherchée, est égale à $l\frac{n}{m}$.

Soit en général une fonction $\int X\partial x$, que X contienne des constantes indéterminées m, n, ... et qu'on cherche des valeurs de $\int X\partial x$, prises depuis $x=A$ jusqu'à $x=B$, la valeur de cette fonction sera égale à l'intégrale de

$$\left(\int\frac{\partial X}{\partial m}\partial B-\int\frac{\partial X}{\partial m}\partial A\right)\partial m+\left(\int\frac{\partial X}{\partial n}\partial B-\int\frac{\partial X}{\partial n}\partial A\right)\partial n+\cdots$$

prise par rapport aux m, n, ...; ainsi, toutes les fois que les fonctions $\frac{\partial X}{\partial m}\partial x$, $\frac{\partial X}{\partial n}\partial x$, . . . seront intégrables, la formule ci-dessus sera débarrassée de signes d'intégration et l'on pourra chercher pour quelles valeurs de A et de B elle devient intégrable.

Sur quoi nous observerons:

1°. Que comme il faut ajouter une arbitraire à cette intégrale, dans le cas où l'on n'auroit pas des moyens de la déterminer, ce ne seroit pas la valeur de $\int X\partial x$, prise depuis $x=A$ jusqu'à $x=B$, qu'on pourroit trouver

par cette méthode, mais celle de $\int(X'\partial x - X\partial x)$, X' étant ce que devient X, en y mettant au lieu de m, n, ... m', n', ...

Par exemple, soit repris l'exemple ci-dessus, où $X = \frac{x^m}{xlx}$, nous avons $\int\frac{x^m}{lx}\cdot\frac{\partial x}{x} = lm + C$; il est clair que lorsque $m = 0$, la valeur de l'intégrale est $ll.1 - ll.0$; or, elle est aussi $l0 + C$; donc, à cause de $l1 = 0$, on a $C = -ll.0$, et

$$\int\frac{x^m}{lx}\frac{\partial x}{x} = lm - ll.0;$$

mais, si on n'avoit pas connu la valeur de l'intégrale pour une particulière de m, C seroit resté inconnu et la méthode n'auroit donné aucun résultat, au lieu que même lorsqu'on ne peut connoître C, elle auroit toujours donné

$$\int\left(\frac{x^{m'}}{xlx} - \frac{x^m}{xlx}\right)\partial x = l\frac{m'}{m}.$$

2°. Que pour trouver, par cette méthode, la valeur de $\int X\partial x$, il faut que, pour satisfaire à l'équation

$$\int\left(\int\frac{\partial X}{\partial m}\partial x\right)\partial m = \int X\partial x,$$

on ne soit pas obligé d'ajouter à la première intégrale, prise par rapport à m, une fonction de x; d'où il résulte qu'il y a encore une infinité de cas où la méthode ne pouvant être employée à trouver $\int X\partial x$, peut l'être à trouver $\int(X' - X)\partial x$.

3°. Que pour réduire l'intégration de $\int X\partial x$ à celle de $\int\left(\int\frac{\partial X}{\partial m}\partial x\right)\partial m$, il suffit de savoir intégrer $\int\frac{\partial X}{\partial m}\partial x$; mais, par la même raison, l'intégration de $\int\frac{\partial X}{\partial m}\partial x$ dépendra de l'intégration de $\int\frac{\partial^2 X}{\partial m^2}\partial x$; en sorte qu'en général, pourvu qu'on puisse trouver $\int\frac{\partial^p X}{\partial m^p}\partial x$, on pourra faire dépendre l'intégration de $\int X\partial x$, d'intégrales prises par rapport à m.

DEMONSTRATION DU SECOND THEOREME

Soit maintenant la fonction

$$(1 + z)^n = 1 + A'z + A''z^2 + A'''z^3 + \cdots$$

en sorte que le coëfficient de z^m soit A^m, nous avons à prouver que

$$1 + A'^2 + A''^2 + A'''^2 + \cdots = \frac{2 \cdot 6 \cdot 10 \cdots (4n-2)}{1 \cdot 2 \cdots n};$$

mettons $n+1$ à la place de n et faisons

$$1 + A'^2 + A''^2 + \cdots = Z,$$

nous aurons

$$\begin{aligned} Z + \varDelta Z = 1 &+ A'^2 + A''^2 + \cdots \\ &+ 2A'\varDelta A' + 2A''\varDelta A'' + \cdots \\ &+ \varDelta A'^2 + \varDelta A''^2 + \cdots \\ &= \frac{2 \cdot 6 \cdot 10 \cdots (4n+2)}{1 \cdot 2 \cdots (n+1)} = \frac{Z(4n+2)}{n+1}; \end{aligned}$$

mais

$$\varDelta A' = 1, \quad \varDelta A'' = A', \quad \varDelta A''' = A'', \ldots$$

donc nous aurons

$$\left.\begin{array}{l} 1 + A'^2 + A''^2 + A'''^2 + \cdots \\ + 2A' + 2A'A'' + 2A''A''' + \cdots \\ + 1 + A'^2 + A''^2 + \cdots \end{array}\right\} = \frac{Z(4n+2)}{n+1}$$

ou

$$2Z + 2(A' + A'A'' + A''A''' + \cdots) = \frac{Z(4n+2)}{n+1},$$

d'où

$$(n+1)(A' + A'A'' + A''A''' + \cdots) = n(1 + A'^2 + A''^2 + A'''^2 + \cdots);$$

or, nous avons

$$A' = n, \quad A'A'' = A'^2 \frac{n-1}{2}, \quad A''A''' = A''^2 \frac{n-2}{3}, \ldots$$

et

$$A' = \frac{A'^2}{n}, \quad A'A'' = A''^2 \frac{2}{n-1}, \quad A''A''' = A'''^2 \frac{3}{n-2}, \ldots$$

Substituant ces valeurs dans l'équation ci-dessus, elle devient

$$\frac{n+1}{n} \left\{ \begin{array}{l} \alpha n + \frac{1-\alpha}{n} A'^2 + \frac{2}{n-1}(1-\alpha')A''^2 + \frac{3}{n-2}(1-\alpha'')A'''^2 + \cdots \\ \qquad + \frac{n-1}{2}\alpha' A'^2 + \frac{n-2}{3}\alpha'' A''^2 + \frac{n-3}{4}\alpha''' A'''^2 + \cdots \end{array} \right\}$$
$$= 1 + A'^2 + A''^2 + A'''^2 + \cdots$$

$a, a', a'', a''' \ldots$ étant des coëfficiens indéterminés; d'où, comparant terme à terme et faisant

$$a = \frac{1}{n+1}, \quad a' = \frac{2}{n+1}, \quad a'' = \frac{3}{n+1}, \quad a''' = \frac{4}{n+1}, \ldots$$

on concluera l'identité des deux formules.

Cette manière d'employer la méthode des coëfficiens indéterminés peut facilement s'étendre à différens théorèmes du même genre.

SUPPLEMENTUM CALCULI INTEGRALIS PRO INTEGRATIONE FORMULARUM IRRATIONALIUM

Commentatio 539 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1780: I (1783), p. 3—31

PROBLEMA 1

1. *Si functio X praeter ipsam variabilem x etiam formulam irrationalem*

$$s = \sqrt{(a + bx)}$$

involvat, ita tamen, ut X sit functio rationalis binarum quantitatum x et s, formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Cum irrationalitas tantum in formula $s = \sqrt{(a + bx)}$ insit, hanc tantum ita per idoneam substitutionem tolli oportet, ut inde valor ipsius x non fiat irrationalis. Hoc autem praestabitur ponendo $a + bx = zz$, ut fiat $s = z$ et

$$x = \frac{zz - a}{b} \quad \text{hincque} \quad dx = \frac{2}{b} zdz;$$

quibus valoribus substitutis tota formula differentialis Xdx ad rationalem, novam variabilem z complectentem, perducitur.

EXEMPLUM 1

2. Si fuerit
$$dy = \frac{dx}{\sqrt{(a+bx)}} \quad \text{seu} \quad dy = \frac{dx}{s},$$
posito $\sqrt{(a+bx)} = z$ fiet
$$dy = \frac{2}{b} dz$$
et integrando $y = \frac{2z}{b}$, unde facta substitutione colligitur
$$y = \frac{2}{b}\sqrt{(a+bx)} + C.$$

EXEMPLUM 2

3. Si fuerit
$$dy = dx\sqrt{(a+bx)} = sdx,$$
sumto $\sqrt{(a+bx)} = z$ erit
$$dy = zdx = \frac{2}{b} zzdz,$$
unde integrando fit $y = \frac{2}{3b} z^3$ et facta substitutione prodit
$$y = \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C.$$
Quod integrale si debeat evanescere facto $x = 0$, fiet $C = -\frac{2a\sqrt{a}}{3b}$ ideoque
$$y = \frac{2(a+bx)^{\frac{3}{2}} - 2a\sqrt{a}}{3b}.$$

EXEMPLUM 3

4. Si fuerit
$$dy = \frac{xdx}{\sqrt{(a+bx)}},$$
facta substitutione $\sqrt{(a+bx)} = z$ erit
$$dy = \frac{2(zz-a)dz}{bb} = \frac{2zzdz - 2adz}{bb},$$

unde fit integrando $y = \frac{2}{3bb}z^3 - \frac{2a}{bb}z + C$ et facta restitutione

$$y = \frac{2}{3bb}(a+bx)^{\frac{3}{2}} - \frac{2a}{bb}\sqrt{(a+bx)} + C = \frac{2\sqrt{(a+bx)}}{bb}\left(\frac{1}{3}bx - \frac{2}{3}a\right) + C.$$

EXEMPLUM 4

5. Si fuerit

$$dy = \frac{dx}{(a+bx)^{\frac{3}{2}}},$$

facta substitutione $\sqrt{(a+bx)} = z$ erit $dy = \frac{dx}{z^3}$; quae formula porro ob $dx = \frac{2zdz}{b}$ abit in

$$dy = \frac{2dz}{bzz},$$

qua integrata fit $y = -\frac{2}{bz}$ seu facta restitutione

$$y = \frac{-2}{b\sqrt{(a+bx)}} + C.$$

Ubi notetur pro C sumi debere $\frac{2}{b\sqrt{a}}$ [si integrale debeat evanescere facto $x = 0$].

PROBLEMA 2

6. *Si fuerit X functio quaecunque rationalis binarum quantitatum x et s existente*

$$s = \sqrt[3]{(a+bx)},$$

formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Ponatur $\sqrt[3]{(a+bx)} = z$, ut sit $s = z$; erit $a + bx = z^3$ hincque

$$x = \frac{z^3 - a}{b} \quad \text{et} \quad dx = \frac{3zzdz}{b};$$

quibus valoribus substitutis tota formula fiet rationalis.

EXEMPLUM 1

7. Si fuerit

$$dy = \frac{dx}{\sqrt[3]{(a+bx)}} = \frac{dx}{s},$$

posito $\sqrt[3]{(a+bx)} = z$ et substituto valore hinc nato $dx = \frac{3zzdz}{b}$ erit

$$dy = \frac{3zdz}{b},$$

unde integrando fit

$$y = \frac{3}{2b}zz = \frac{3}{2b}\sqrt[3]{(a+bx)^2} + C.$$

EXEMPLUM 2

8. Si fuerit

$$dy = \frac{dx}{\sqrt[3]{(a+bx)^2}} = \frac{dx}{ss},$$

posito $\sqrt[3]{(a+bx)} = z$ fiet

$$dy = \frac{3dz}{b},$$

hinc integrando

$$y = \frac{3}{b}z = \frac{3}{b}\sqrt[3]{(a+bx)} + C.$$

EXEMPLUM 3

9. Si fuerit

$$dy = dx\sqrt[3]{(a+bx)} = sdx,$$

facta substitutione fit

$$dy = \frac{3z^3dz}{b},$$

hinc integrando

$$y = \frac{3}{4b}z^4 = \frac{3}{4b}(a+bx)\sqrt[3]{(a+bx)} + C.$$

PROBLEMA 3

10. *Si fuerit X functio rationalis binarum quantitatum x et s existente*

$$s = \sqrt[n]{(a+bx)},$$

formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Ponatur $\sqrt[n]{(a+bx)} = z$, ut sit $s = z$; erit $a + bx = z^n$, hinc

$$x = \frac{z^n - a}{b} \quad \text{et} \quad dx = \frac{nz^{n-1}dz}{b};$$

quibus valoribus substitutis formula proposita Xdx certe fiet rationalis, si modo numerus exponentialis n fuerit integer.

EXEMPLUM 1

11. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{(a+bx)}} = \frac{dx}{s},$$

posito $\sqrt[n]{(a+bx)} = z$ ob valorem inde natum $dx = \frac{nz^{n-1}dz}{b}$ habebitur

$$dy = \frac{nz^{n-2}dz}{b},$$

unde integrando colligimus $y = \frac{n}{b(n-1)}z^{n-1} + C$ sive restitutis valoribus

$$y = \frac{n}{b(n-1)}(a+bx)^{\frac{n-1}{n}} + C = \frac{n}{b(n-1)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)}} + C.$$

EXEMPLUM 2

12. Si fuerit

$$dy = \frac{dx}{\sqrt[n]{(a+bx)^\lambda}} = \frac{dx}{s^\lambda},$$

posito $\sqrt[n]{(a+bx)} = z$ et substituto valore $dx = \frac{nz^{n-1}dz}{b}$ fiet

$$dy = \frac{nz^{n-1}dz}{bz^{\lambda}} = \frac{n}{b}z^{n-\lambda-1}dz,$$

cuius integrale dat

$$y = \frac{n}{b(n-\lambda)}(a+bx)^{\frac{n-\lambda}{n}} + C \quad \text{sive} \quad y = \frac{n}{b(n-\lambda)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)^{\lambda}}} + C.$$

Ex his autem exemplis iam apparet integrationem non impediri, etiamsi exponentes n et λ non fuerint numeri integri.

PROBLEMA 4

13. *Si fuerit X functio rationalis binarum quantitatum x et s existente*

$$s = \sqrt{\left(a + b\sqrt{(f+gx)}\right)},$$

quae formula ergo duplicem irrationalitatem involvit, formulam differentialem Xdx ab hac duplici irrationalitate liberare.

SOLUTIO

Ponatur iterum $\sqrt{\left(a+b\sqrt{(f+gx)}\right)} = z$, ut sit $s = z$; erit sumtis quadratis $a + b\sqrt{(f+gx)} = zz$, hinc $b\sqrt{(f+gx)} = zz - a$ ac sumtis denuo quadratis $bb(f+gx) = (zz-a)^2$, unde colligitur

$$x = \frac{(zz-a)^2}{bbg} - \frac{f}{g} \quad \text{hincque} \quad dx = \frac{4zdz(zz-a)}{bbg}.$$

Quibus valoribus substitutis tota formula reddetur rationalis.

COROLLARIUM

14. Perspicuum est eodem modo irrationalitatem tolli posse, si fuerit multo generalius

$$s = \sqrt[n]{\left(a + b\sqrt[m]{(f+gx)}\right)}.$$

Posita enim hac formula $= z$ fiet

$$a + b\sqrt[m]{(f + gx)} = z^n \quad \text{et} \quad b\sqrt[m]{(f + gx)} = z^n - a.$$

Porro $b^m(f + gx) = (z^n - a)^m$ et hinc colligitur

$$x = \frac{(z^n - a)^m}{b^m g} - \frac{f}{g} \quad \text{ideoque} \quad dx = \frac{mnz^{n-1}dz(z^n - a)^{m-1}}{b^m g}.$$

Sicque etiam hoc modo tota formula rationalis evadet.

PROBLEMA 5

15. *Si fuerit X functio rationalis binarum quantitatum x et s existente*

$$s = \sqrt{\frac{a + bx}{f + gx}},$$

formulam differentialem Xdx ab irrationalitate liberare.

SOLUTIO

Ponatur $\sqrt{\frac{a+bx}{f+gx}} = z$ et sumtis quadratis erit $\frac{a+bx}{f+gx} = zz$ hincque

$$x = \frac{fzz - a}{b - gzz},$$

unde differentiando colligitur

$$dx = \frac{2bfzdz - 2agzdz}{(b - gzz)^2}.$$

Hisque valoribus substitutis formula proposita Xdx ad rationalitatem erit perducta.

EXEMPLUM 1

16. Si fuerit

$$dy = \frac{dx}{s} = \frac{dx\sqrt{(f + gx)}}{\sqrt{(a + bx)}},$$

posito $\sqrt{\frac{a+bx}{f+gx}}=z$ erit $dy=\frac{dx}{z}$ et substituto loco dx valore supra invento colligitur

$$dy=\frac{2(bf-ag)\,dz}{(b-gzz)^2};$$

quae formula, uti iam satis constat, reduci potest ad talem

$$\int\frac{dz}{b-gzz},$$

cuius autem integratio vel per logarithmos vel per arcus circulares expedietur.

EXEMPLUM 2

17. Sit specialius

$$dy=\frac{dx\sqrt{(1-x)}}{\sqrt{(1+x)}},$$

ubi $f=1$, $g=-1$, $a=1$ et $b=1$ ideoque

$$z=\frac{\sqrt{(1+x)}}{\sqrt{(1-x)}}\ \text{ et }\ dx=\frac{4z\,dz}{(1+zz)^2};$$

quibus valoribus substitutis fiet

$$dy=\frac{4\,dz}{(1+zz)^2}.$$

Statuatur ergo

$$\int\frac{4\,dz}{(1+zz)^2}=\frac{Az}{1+zz}+B\int\frac{dz}{1+zz}=y,$$

unde sumtis differentialibus fiet

$$\frac{4}{(1+zz)^2}=\frac{A-Azz}{(1+zz)^2}+\frac{B}{1+zz}=\frac{A+B+(B-A)zz}{(1+zz)^2}.$$

Oportet igitur esse $A+B=4$ et $B-A=0$ ideoque $A=2$ et $B=2$; et quia $\int\frac{dz}{1+zz}=\text{A tang. } z$, adipiscimur $y=\frac{2z}{1+zz}+2\text{ A tang. } z$, quocirca facta restitutione ob $1+zz=\frac{2}{1-x}$ obtinebitur

$$y=\sqrt{(1-xx)}+2\text{ A tang. }\sqrt{\frac{1+x}{1-x}}.$$

Cum igitur huius arcus tangens sit $\sqrt{\frac{1+x}{1-x}}$, erit eius sinus $=\sqrt{\frac{1+x}{2}}$ et cosinus $=\sqrt{\frac{1-x}{2}}$; anguli vero dupli sinus erit $\sqrt{(1-xx)}$ et cosinus $=-x$, unde fiet

$$2\,\mathrm{A\ tang.}\sqrt{\frac{1+x}{1-x}}=\mathrm{A\ cos.}-x=\frac{\pi}{2}+\mathrm{A\ sin.}\,x;$$

quocirca integrale quaesitum erit

$$y=\sqrt{(1-xx)}+\frac{\pi}{2}+\mathrm{A\ sin.}\,x+C;$$

quod si ita capi debeat, ut evanescat posito $x=0$, erit $C=-1-\frac{\pi}{2}$ ideoque

$$y=\sqrt{(1-xx)}-1+\mathrm{A\ sin.}\,x.$$

Tum igitur si sumatur $x=1$, fiet $y=\frac{\pi}{2}-1$, qui valor in fractionibus decimalibus dat 0,5707963.

PROBLEMA 6

18. *Si fuerit X functio rationalis binarum variabilium x et s existente*

$$s=\sqrt[n]{\frac{a+bx}{f+gx}},$$

formulam differentialem Xdx ad rationalitatem perducere.

SOLUTIO

Posito $s=\sqrt[n]{\frac{a+bx}{f+gx}}=z$ erit $\frac{a+bx}{f+gx}=z^n$ hincque

$$x=\frac{fz^n-a}{b-gz^n},$$

consequenter

$$dx=\frac{n(bf-ag)z^{n-1}dz}{(b-gz^n)^2};$$

hisque valoribus substitutis tota formula proposita Xdx ad rationalitatem erit perducta.

PROBLEMA 7

19. *Si fuerit X functio binarum quantitatum xx et s existente*

$$s = \sqrt{(a + bxx)},$$

formulam differentialem $\frac{Xdx}{x}$ ab irrationalitate liberare.

SOLUTIO

Ponamus $s = \sqrt{(a + bxx)} = z$; erit $a + bxx = zz$, hinc

$$xx = \frac{zz - a}{b},$$

et quia in functione X tantum quadratum xx eiusque ergo potestates pares occurrunt, hac substitutione iam functio X evadet rationalis. Sumtis vero logarithmis $2lx = l(zz - a) - lb$ differentiando fit $\frac{2dx}{x} = \frac{2zdz}{zz - a}$ ideoque

$$\frac{dx}{x} = \frac{zdz}{zz - a}.$$

Hoc ergo modo formula proposita $X\frac{dx}{x}$ prorsus reddetur rationalis.

EXEMPLUM 1

20. Si fuerit

$$dy = \frac{xdx}{\sqrt{(a + bxx)}},$$

erit

$$dy = \frac{dx}{x} \cdot \frac{xx}{\sqrt{(a + bxx)}} = \frac{xx}{s} \cdot \frac{dx}{x}.$$

Posito ergo $\sqrt{(a + bxx)} = z$ erit

$$dy = \frac{dz}{b},$$

unde colligitur integrando

$$y = \frac{z}{b} = \frac{\sqrt{(a + bxx)}}{b}.$$

EXEMPLUM 2

21. Si fuerit

$$dy = \frac{x^3 dx}{\sqrt{(a+bxx)}} = \frac{dx}{x} \cdot \frac{x^4}{s},$$

ponendo $\sqrt{(a+bxx)} = z$, ut sit

$$xx = \frac{zz-a}{b} \quad \text{et} \quad \frac{dx}{x} = \frac{z\,dz}{zz-a},$$

erit

$$dy = \frac{1}{bb} dz(zz-a)$$

hincque integrando adipiscimur $y = \frac{z}{3bb}(zz-3a)$; unde facta restitutione prodibit integrale quaesitum

$$y = \frac{bxx-2a}{3bb}\sqrt{(a+bxx)} + C.$$

EXEMPLUM 3

22. Si fuerit

$$dy = \frac{x^3 dx}{\sqrt{(a+bxx)^3}},$$

erit $dy = \frac{dx}{x} \cdot \frac{x^4}{s^3}$; hinc posito $\sqrt{(a+bxx)} = s = z$ fiet

$$dy = \frac{dz}{bb} \cdot \frac{zz-a}{zz},$$

unde sumto integrali fiet $y = \frac{1}{bb} \cdot \frac{zz+a}{z}$, quocirca facta restitutione resultat

$$y = \frac{2a+bxx}{bb\sqrt{(a+bxx)}} + C.$$

PROBLEMA 8

23. *Si fuerit* X *functio rationalis binarum quantitatum* x^n *et* s *existente*

$$s = \sqrt[m]{(a+bx^n)},$$

formulam differentialem $X\frac{dx}{x}$ *ad rationalitatem perducere.*

SOLUTIO

Posito $s = \sqrt[m]{(a + bx^n)} = z$ fiet $a + bx^n = z^m$ et

$$x^n = \frac{z^m - a}{b}.$$

Quia igitur in functione X tantum potestas x^n occurrit, ea rationalis reddetur, si hi valores substituantur. Tum vero sumtis logarithmis habebitur

$$nlx = l(z^m - a) - lb$$

et differentiando

$$\frac{dx}{x} = \frac{mz^{m-1}dz}{n(z^m - a)};$$

sicque tota formula proposita fiet rationalis.

EXEMPLUM

24. Sit

$$dy = \frac{x^{n-1}dx}{\sqrt[m]{(a + bx^n)}} = \frac{dx}{x} \cdot \frac{x^n}{s}$$

factaque substitutione orietur haec aequatio

$$dy = \frac{mz^{m-2}dz}{bn},$$

qua integrata prodibit

$$y = \frac{mz^{m-1}}{nb(m-1)} = \frac{m}{nb(m-1)} \sqrt[m]{(a + bx^n)^{m-1}} + C$$

sive

$$y = \frac{m}{nb(m-1)} \cdot \frac{a + bx^n}{\sqrt[m]{(a + bx^n)}} + C.$$

PROBLEMA 9

25. *Si fuerit X functio rationalis quantitatum xx et s existente*

$$s = \sqrt{\frac{a + bxx}{f + gxx}},$$

formulam differentialem $X\frac{dx}{x}$ ab irrationalitate liberare.

SOLUTIO

Ponatur $s = \sqrt{\frac{a+bxx}{f+gxx}} = z$ eritque $\frac{a+bxx}{f+gxx} = zz$, hinc

$$xx = \frac{fzz-a}{b-gzz},$$

unde functio X penitus fit rationalis. Porro sumtis logarithmis

$$2lx = l(fzz-a) - l(b-gzz)$$

differentietur, ut prodeat

$$\frac{2dx}{x} = \frac{2fzdz}{fzz-a} + \frac{2gzdz}{b-gzz} = \frac{2(bf-ag)zdz}{(fzz-a)(b-gzz)},$$

unde fit

$$\frac{dx}{x} = \frac{(bf-ag)zdz}{(fzz-a)(b-gzz)};$$

sicque tota formula differentialis fiet rationalis.

EXEMPLUM

26. Si fuerit

$$dy = \frac{dx}{\sqrt{(f+gxx)}},$$

repraesentemus hanc formulam ita

$$dy = \frac{dx}{x}\cdot\frac{x}{\sqrt{(f+gxx)}} = \frac{dx}{x}\sqrt{\frac{xx}{f+gxx}}.$$

Hic ergo erit $a=0$, $b=1$ et $z = \frac{x}{\sqrt{(f+gxx)}}$, ita ut $dy = \frac{zdx}{x}$; erit autem $\frac{dx}{x} = \frac{dz}{z(1-gzz)}$, unde fit

$$dy = \frac{dz}{1-gzz},$$

cuius formulae integratio per logarithmos expedietur, si fuerit g numerus positivus; sin autem fuerit negativus, per arcus circulares absolvetur.

Sit igitur 1° $g = +bb$; erit

$$dy = \frac{dz}{1-bbzz}$$

ideoque

$$y = \frac{1}{2b}l\frac{1+bz}{1-bz}$$

et restitutis valoribus supra indicatis erit

$$y = \frac{1}{2b} l \frac{\sqrt{(f+bbxx)}+bx}{\sqrt{(f+bbxx)}-bx} = \frac{1}{b} l \frac{\sqrt{(f+bbxx)}+bx}{\sqrt{f}}.$$

Sit 2^0 g quantitas negativa, puta $g = -bb$; erit

$$dy = \frac{dz}{1+bbzz} = \frac{1}{b} \cdot \frac{bdz}{1+bbzz},$$

unde colligitur

$$y = \frac{1}{b} \text{A tang. } bz = \frac{1}{b} \text{A tang. } \frac{bx}{\sqrt{(f-bbxx)}}.$$

Ubi manifestum est f esse debere quantitatem positivam, quia alioquin formula differentialis esset imaginaria.

COROLLARIUM

27. Hinc ergo, si proponatur formula

$$dy = \frac{dx}{\sqrt{(1+xx)}},$$

ubi $f=1$ et $g=1$, ex casu priore ob $b=+1$ erit

$$\int \frac{dx}{\sqrt{(1+xx)}} = l\left(\sqrt{(1+xx)}+x\right).$$

At si fuerit

$$dy = \frac{dx}{\sqrt{(1-xx)}},$$

ubi $f=1$ et $g=-1$, colligitur ex casu posteriore $y = \text{A tang. } \frac{x}{\sqrt{(1-xx)}}$, unde concluditur

$$\int \frac{dx}{\sqrt{(1-xx)}} = \text{A sin. } x = \text{A cos. } \sqrt{(1-xx)}.$$

PROBLEMA 10

28. *Si fuerit X functio rationalis quantitatum x^n et s existente*

$$s = \sqrt[n]{\frac{a+bx^n}{f+gx^n}},$$

formulam differentialem $X\frac{dx}{x}$ rationalem efficere.

SOLUTIO

Ponatur $s = \sqrt[n]{\frac{a+bx^n}{f+gx^n}} = z$ eritque $\frac{a+bx^n}{f+gx^n} = z^n$, hinc

$$x^n = \frac{fz^n - a}{b - gz^n};$$

tum autem sumtis logarithmis erit

$$nlx = l(fz^n - a) - l(b - gz^n)$$

et differentiando

$$\frac{dx}{x} = \frac{fz^{n-1}dz}{fz^n - a} + \frac{gz^{n-1}dz}{b - gz^n} = \frac{(bf - ag)z^{n-1}dz}{(fz^n - a)(b - gz^n)};$$

quibus valoribus substitutis formula proposita fit rationalis.

PROBLEMA 11

29. *Si fuerit X functio rationalis binarum quantitatum x^n et s existente*

$$s = \sqrt[m]{\frac{a+bx^n}{f+gx^n}},$$

formulam differentialem $X\frac{dx}{x}$ ab omni irrationalitate liberare.

SOLUTIO

Statuatur $s = \sqrt[m]{\frac{a+bx^n}{f+gx^n}} = z$ eritque $\frac{a+bx^n}{f+gx^n} = z^m$, unde fit

$$x^n = \frac{fz^m - a}{b - gz^m};$$

hinc sumtis logarithmis erit

$$nlx = l(fz^m - a) - l(b - gz^m),$$

hinc differentiando

$$\frac{ndx}{x} = \frac{m(bf - ag)z^{m-1}dz}{(fz^m - a)(b - gz^m)}$$

ideoque

$$\frac{dx}{x} = \frac{m(bf - ag)z^{m-1}dz}{n(fz^m - a)(b - gz^m)};$$

quibus valoribus substitutis irrationalitas formulae propositae penitus tollitur.

PROBLEMA 12

30. *Si fuerit X functio rationalis quaecunque binarum quantitatum x et s existente*

$$s = \sqrt{(\alpha + \beta x + \gamma xx)},$$

formulam differentialem Xdx ad rationalitatem perducere.

SOLUTIO

Hic duos casus a se invicem distingui convenit, prout γ fuerit vel quantitas positiva vel negativa.

I. Sit γ quantitas positiva ac ponatur $\gamma = cc$ et $\beta = 2bc$, ut habeatur

$$s = \sqrt{(\alpha + 2bcx + ccxx)} = \sqrt{(\alpha - bb + (b + cx)^2)},$$

ubi loco $\alpha - bb$ brevitatis ergo scribatur e, ut sit $s = \sqrt{(e + (b + cx)^2)}$. Iam statuatur $s = b + cx + z$ eritque

$$ss = e + (b + cx)^2 = (b + cx)^2 + 2(b + cx)z + zz,$$

unde sequitur

$$e - zz = 2z(b + cx) \quad \text{sive} \quad b + cx = \frac{e - zz}{2z};$$

hincque colligitur

$$x = \frac{e - zz}{2cz} - \frac{b}{c} \quad \text{seu} \quad x = \frac{e - 2bz - zz}{2cz}.$$

Aequatio autem $b + cx = \frac{e - zz}{2z}$ differentiata praebet

$$cdx = -\frac{edz}{2zz} - \frac{dz}{2} = -\frac{edz + zzdz}{2zz},$$

unde deducitur

$$dx = -\frac{dz(e + zz)}{2czz};$$

at ob $b + cx = \frac{e - zz}{2z}$ fiet

$$s = \frac{e + zz}{2z}.$$

His ergo valoribus substitutis formula nostra Xdx reddetur rationalis. Postquam igitur eius integrale fuerit inventum, loco z valor ante inventus $\sqrt{(e + (b + cx)^2)} - b - cx$ erit substituendus.

II. Sin autem γ fuerit quantitas negativa, ponatur $\gamma = -cc$ et $\beta = -2bc$, ut habeatur

$$s = \sqrt{(\alpha - 2bcx - ccxx)} = \sqrt{(\alpha + bb - (b + cx)^2)},$$

ubi evidens est quantitatem $\alpha + bb$ necessario esse debere positivam, quia alioquin s evaderet imaginarium. Quamobrem ponamus brevitatis gratia $\alpha + bb = aa$, ut fiat $s = \sqrt{(aa - (b + cx)^2)}$, ad quam formam rationalem efficiendam statuamus

$$\sqrt{(aa - (b + cx)^2)} = a - (b + cx)z,$$

unde sumtis quadratis erit

$$aa - (b + cx)^2 = aa - 2az(b + cx) + (b + cx)^2 zz,$$

quae aequatio reducitur ad hanc

$$-(b + cx) = -2az + (b + cx)zz,$$

unde reperitur

$$b + cx = \frac{2az}{1 + zz}$$

ideoque

$$x = \frac{2az - b - bzz}{c(1 + zz)}.$$

Illa autem aequatio differentiata dat

$$cdx = \frac{2adz(1 + zz) - 4azzdz}{(1 + zz)^2} = \frac{2adz(1 - zz)}{(1 + zz)^2},$$

unde fit

$$dx = \frac{2adz(1 - zz)}{c(1 + zz)^2}.$$

Porro autem cum sit $s = a - (b + cx)z$, ob $b + cx = \frac{2az}{1 + zz}$ erit

$$s = \frac{a(1 - zz)}{1 + zz},$$

quocirca, si loco x, s et dx inventi hi valores substituantur, formula proposita differentialis Xdx evadet rationalis et per variabilem z exprimetur; cuius integrale postquam fuerit inventum, loco z ubique eius restituatur valor assumtus

$$z = \frac{a - \sqrt{(aa - (b + cx)^2)}}{b + cx}$$

et integrale obtinebitur per solam variabilem x expressum.

EXEMPLUM 1

31. Si fuerit

$$dy = \frac{dx}{\sqrt{(e+(b+cx)^2)}},$$

quae formula ad casum priorem pertinet, erit

$$dy = \frac{dx}{s} = -\frac{dz}{cz}$$

ob

$$dx = -\frac{dz(e+zz)}{2czz} \quad \text{et} \quad s = \frac{e+zz}{2z},$$

cuius integrale est $y = -\frac{1}{c} lz$; restituto ergo valore

$$z = \sqrt{(e+(b+cx)^2)} - b - cx$$

erit

$$y = -\frac{1}{c} l\left(\sqrt{(e+(b+cx)^2)} - b - cx\right) + C;$$

quod integrale si evanescere debeat posito $x = 0$, fiet

$$C = \frac{1}{c} l\left(\sqrt{(e+bb)} - b\right).$$

COROLLARIUM

32. Si ponatur $b = 0$ et $c = 1$ sive

$$dy = \frac{dx}{\sqrt{(e+xx)}},$$

erit integrale

$$y = -l\left(\sqrt{(e+xx)} - x\right) + l\sqrt{e} = l\frac{\sqrt{e}}{\sqrt{(e+xx)} - x},$$

quae formula reducitur ad hanc

$$y = l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}}.$$

Cum vero porro sit $d.\sqrt{(e+xx)} = \frac{xdx}{\sqrt{(e+xx)}}$, erit

$$\int \frac{xdx}{\sqrt{(e+xx)}} = \sqrt{(e+xx)}.$$

Si igitur hae duae formulae combinentur, habebitur ista integratio notatu digna

$$\int \frac{A\,dx + Bx\,dx}{\sqrt{(e+xx)}} = A\,l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}} + B\sqrt{(e+xx)}.$$

EXEMPLUM 2

33. Sit

$$dy = \frac{dx}{\sqrt{(aa-(b+cx)^2)}},$$

quae formula ad casum secundum est referenda, ita ut sit $dy = \frac{dx}{s}$. Cum igitur sit

$$dx = \frac{2a\,dz(1-zz)}{c(1+zz)^2} \quad \text{et} \quad s = \frac{a(1-zz)}{1+zz},$$

erit

$$dy = \frac{dx}{s} = \frac{2}{c}\cdot\frac{dz}{1+zz},$$

unde fit integrando $y = \frac{2}{c}\,\mathrm{A\,tang.}\,z$. Quia igitur est

$$z = \frac{a-\sqrt{(aa-(b+cx)^2)}}{b+cx},$$

erit

$$y = \frac{2}{c}\,\mathrm{A\,tang.}\,\frac{a-\sqrt{(aa-(b+cx)^2)}}{b+cx} + C.$$

COROLLARIUM

34. Sit igitur iterum $b=0$ et $c=1$ seu formula differentialis proposita

$$dy = \frac{dx}{\sqrt{(aa-xx)}}$$

reperieturque

$$y = 2\,\mathrm{A\,tang.}\frac{a-\sqrt{(aa-xx)}}{x} + C.$$

Quia igitur tangens huius arcus est $\frac{a-\sqrt{(aa-xx)}}{x}$, tangens dupli arcus erit $\frac{x}{\sqrt{(aa-xx)}}$, ita ut sit

$$y = \mathrm{A\,tang.}\,\frac{x}{\sqrt{(aa-xx)}};$$

huius autem arcus sinus erit $\frac{x}{a}$ sicque integrale quaesitum

$$\int \frac{dx}{\sqrt{(aa-xx)}} = \mathrm{A}\ \sin. \frac{x}{a}.$$

Quia porro $d.\sqrt{(aa-xx)} = -\frac{xdx}{\sqrt{(aa-xx)}}$, erit

$$\int \frac{xdx}{\sqrt{(aa-xx)}} = -\sqrt{(aa-xx)},$$

quocirca ista generalior conficitur integratio

$$\int \frac{Adx + Bxdx}{\sqrt{(aa-xx)}} = A\mathrm{A}\ \sin. \frac{x}{a} - B\sqrt{(aa-xx)}.$$

PROBLEMA 13

35. *Si fuerit V functio rationalis binarum quantitatum v^n et s existente*

$$s = \sqrt{(\alpha + \beta v^n + \gamma v^{2n})},$$

formulam differentialem $Vv^{n-1}dv$ ab irrationalitate liberare.

SOLUTIO

Ponatur $v^n = x$; erit

$$s = \sqrt{(\alpha + \beta x + xx)} \quad \text{et} \quad v^{n-1}dv = \frac{dx}{n};$$

hic ergo iam erit V functio rationalis binarum quantitatum x et s existente $s = \sqrt{(\alpha + \beta x + \gamma xx)}$ et formula ab irrationalitate liberanda erit $\frac{Vdx}{n}$; qui casus prorsus convenit cum problemate praecedente ideoque eandem habebit solutionem.

SCHOLION

36. Praecepta hactenus tradita ad omnes fere formulas differentiales, quae quidem adhuc tractari potuerunt, extenduntur. Interim tamen eiusmodi casus occurrere possunt, quibus idonea substitutio ad irrationalitatem tollendam neccessaria non tam facile perspicitur, sed acri iudicio demum investigari licet; in quo negotio cum praecepta generalia tradere nondum liceat, exempla quaedam particularia speciminis loco in medium afferamus.

EXEMPLUM 1

37. *Si proposita fuerit haec formula irrationalis*

$$dP = \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

eius integrale P investigare.

Si quis hic eiusmodi uti vellet substitutione, qua formula $\sqrt{(1+x^4)}$ ad rationalitatem perduceretur, oleum et operam esset perditurus; interim tamen singulari artificio sequens substitutio negotium conficere poterit. Statuatur

$$\frac{x\sqrt{2}}{1-xx} = p$$

eritque $1+pp = \frac{1+x^4}{(1-xx)^2}$, hinc

$$\sqrt{(1+pp)} = \frac{\sqrt{(1+x^4)}}{1-xx};$$

tum vero erit differentiando

$$dp = \frac{dx\sqrt{2}(1+xx)}{(1-xx)^2},$$

ex quibus valoribus colligitur

$$\frac{dp}{\sqrt{(1+pp)}} = \frac{dx\sqrt{2}(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

quae feliciter cum formula ipsa proposita convenit, ita ut sit

$$\frac{dp}{\sqrt{(1+pp)}} = dP\sqrt{2} \quad \text{sive} \quad dP = \frac{1}{\sqrt{2}}\cdot\frac{dp}{\sqrt{(1+pp)}},$$

unde colligitur integrando

$$P = \frac{1}{\sqrt{2}}\, l\left(\sqrt{(1+pp)}+p\right).$$

Quare si loco p et $\sqrt{(1+pp)}$ valores dati substituantur, haec obtinetur integratio satis memorabilis

$$P = \int\frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\, l\,\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx}.$$

EXEMPLUM 2

38. *Si proposita fuerit haec formula irrationalis*

$$\frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

eius integrale Q investigare.

Ad hoc praestandum fiat

$$\frac{x\sqrt{2}}{1+xx}=q$$

eritque

$$\sqrt{(1-qq)}=\frac{\sqrt{(1+x^4)}}{1+xx};$$

tum vero erit

$$dq=\frac{dx(1-xx)\sqrt{2}}{(1+xx)^2}$$

atque hinc colligitur

$$\frac{dq}{\sqrt{(1-qq)}}=\frac{dx(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}=dQ\sqrt{2},$$

unde fit

$$Q=\frac{1}{\sqrt{2}}\int\frac{dq}{\sqrt{(1-qq)}}=\frac{1}{\sqrt{2}}\,\mathrm{A}\sin.q.$$

Restituto ergo pro q valore assumto ista obtinebitur integratio

$$Q=\int\frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}}=\frac{1}{\sqrt{2}}\,\mathrm{A}\sin.\frac{x\sqrt{2}}{1+xx}.$$

SCHOLION

39. Cum istae duae formulae

$$\frac{dx(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}}\quad\text{et}\quad\frac{dx(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}$$

perductae sint ad has simplices

$$\frac{dp}{\sqrt{(1+pp)}}\quad\text{et}\quad\frac{dq}{\sqrt{(1-qq)}},$$

quarum utraque facile ab irrationalitate liberatur, istae ipsae formulae propositae ope idoneae substitutionis ab irrationalitate liberari possunt; unde mirum non est earum integralia sive per logarithmum sive per arcum circularem exhiberi potuisse. Satis enim iam est ostensum omnium formularum differentialium rationalium integralia semper vel per logarithmos et arcus circulares vel adeo algebraice exhiberi posse; quod igitur etiam de illis formulis irrationalibus est tenendum, quas certae substitutionis ope ad rationalitatem perducere licet. Unde vicissim plures Geometrae concluserunt, si quae formula differentialis nullo plane modo ab irrationalitate liberari queat, tum eius integrale etiam neque per logarithmos nec arcus circulares, multo minus algebraice exprimi posse, sed ad aliud genus quantitatum transcendentium referri oportere. Ceterum combinatio duorum praecedentium exemplorum manuducit ad solutionem sequentium.

EXEMPLUM 3

40. *Si proposita fuerit haec formula differentialis*

$$dy = \frac{dx\sqrt{(1+x^4)}}{1-x^4},$$

eius integrale invenire.

Hanc formulam per neutram substitutionem ante usurpatam rationalem reddere licet, utraque tamen iuncta negotium confici poterit; namque eius integrale per logarithmos et arcus circulares sequenti artificio expedietur. Formula enim proposita in binas sequentes partes discerpi potest, quae sunt

$$dy = \frac{\frac{1}{2}dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} + \frac{\frac{1}{2}dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quarum summa ipsam formulam nostram propositam producit; prodit enim

$$dy = \frac{\frac{1}{2}dx(1+xx)^2 + \frac{1}{2}dx(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}} = \frac{dx(1+x^4)}{(1-x^4)\sqrt{(1+x^4)}} = \frac{dx\sqrt{(1+x^4)}}{1-x^4}.$$

Quodsi ergo duo praecedentia exempla in subsidium vocentur, manifesto fiet $dy = \frac{1}{2}dP + \frac{1}{2}dQ$, consequenter integrale quaesitum erit $y = \frac{1}{2}P + \frac{1}{2}Q$,

quod sequenti modo exprimere licebit

$$\int \frac{dx\sqrt{(1+x^4)}}{1-x^4} = \frac{1}{2\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} + \frac{1}{2\sqrt{2}}\, \mathrm{A\,sin.} \frac{x\sqrt{2}}{1+xx}.$$

EXEMPLUM 4

41. *Si proposita fuerit haec formula differentialis*

$$dy = \frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}},$$

eius integrale investigare.

Haec formula simili modo ac praecedens tractari potest; discerpatur enim in sequentes duas partes

$$\frac{\frac{1}{4}dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} - \frac{\frac{1}{4}dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quae coniunctae producunt

$$dy = \frac{\frac{1}{4}dx(1+xx)^2 - \frac{1}{4}dx(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}} = \frac{\frac{1}{4}dx \,.\, 4xx}{(1-x^4)\sqrt{(1+x^4)}} = \frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}};$$

quae cum sit ipsa formula proposita, erit ex praecedentibus exemplis $dy = \frac{1}{4}dP - \frac{1}{4}dQ$, consequenter $y = \frac{1}{4}P - \frac{1}{4}Q$, hinc integrale quaesitum ita reperietur expressum

$$\int \frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} - \frac{1}{4\sqrt{2}}\, \mathrm{A\,sin.} \frac{x\sqrt{2}}{1+xx}.$$

SCHOLION

42. Haec duo postrema exempla si nullo plane modo ope cuiuspiam substitutionis ad rationalitatem perduci possent, insigne praeberent documentum, quod conclusio supra memorata quandoque fallere possit. Re autem attentius perpensa inveni omnia haec quatuor exempla ope unicae substitutionis immediate ad rationalitatem perduci ideoque integrari posse; id quod ostendisse utique operae erit pretium.

ALIA RESOLUTIO QUATUOR POSTREMORUM EXEMPLORUM

43. Statuatur pro primo exemplo

$$v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$$

eritque

$$\sqrt{(1+vv)} = \frac{1+xx}{\sqrt{(1+x^4)}};$$

tum vero

$$\sqrt{(1-vv)} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

unde fit

$$\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx} \quad \text{et} \quad \sqrt{(1-v^4)} = \frac{1-x^4}{1+x^4}.$$

At differentiando adipiscimur

$$dv = \frac{dx(1-x^4)\sqrt{2}}{(1+x^4)\sqrt{(1+x^4)}}.$$

Cum nunc sit $\frac{1-x^4}{1+x^4} = \sqrt{(1-v^4)}$, erit $dv = \frac{dx\sqrt{2}\sqrt{(1-v^4)}}{\sqrt{(1+x^4)}}$ sive

$$\frac{dv}{\sqrt{(1-v^4)}} = \frac{dx\sqrt{2}}{\sqrt{(1+x^4)}},$$

quae aequalitas maxime est notatu digna. Quodsi iam haec aequatio multiplicetur per $\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx}$, nascetur haec aequatio

$$\frac{dv}{1-vv} = \frac{dx(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}}$$

sicque erit

$$\int \frac{dx(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\int \frac{dv}{1-vv} = \frac{1}{2\sqrt{2}}\, l\frac{1+v}{1-v}.$$

Deinde aequatio

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{\sqrt{(1-v^4)}} = \frac{dx}{\sqrt{(1+x^4)}}$$

multiplicetur per $\sqrt{\frac{1-vv}{1+vv}} = \frac{1-xx}{1+xx}$ ac prodibit formula exempli secundi

$$\int \frac{dx(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\int \frac{dv}{1+vv} = \frac{1}{\sqrt{2}} \text{A tang.}\, v.$$

Porro eadem aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{dv}{\sqrt{(1-v^4)}}=\frac{dx}{\sqrt{(1+x^4)}}$$

dividatur per $\sqrt{(1-v^4)}=\frac{1-x^4}{1+x^4}$ et prodibit

$$\frac{1}{\sqrt{2}}\cdot\frac{dv}{1-v^4}=\frac{dx\sqrt{(1+x^4)}}{1-x^4};$$

quae est ipsa formula exempli tertii, ita ut iam sit

$$\int\frac{dx\sqrt{(1+x^4)}}{1-x^4}=\frac{1}{\sqrt{2}}\int\frac{dv}{1-v^4}=\frac{1}{2\sqrt{2}}\int\frac{dv}{1+vv}+\frac{1}{2\sqrt{2}}\int\frac{dv}{1-vv},$$

quod integrale cum ante invento egregie convenit. Tandem postrema aequatio hic inventa

$$\frac{1}{\sqrt{2}}\cdot\frac{dv}{1-v^4}=\frac{dx\sqrt{(1+x^4)}}{1-x^4}$$

ducatur in $vv=\frac{2xx}{1+x^4}$, ut prodeat

$$\frac{1}{\sqrt{2}}\cdot\frac{vvdv}{1-v^4}=\frac{2xxdx\sqrt{(1+x^4)}}{(1-x^4)(1+x^4)}=\frac{2xxdx}{(1-x^4)\sqrt{(1+x^4)}},$$

unde pro exemplo quarto colligitur

$$\int\frac{xxdx}{(1-x^4)\sqrt{(1+x^4)}}=\frac{1}{2\sqrt{2}}\int\frac{vvdv}{1-v^4}=-\frac{1}{4\sqrt{2}}\int\frac{dv}{1+vv}+\frac{1}{4\sqrt{2}}\int\frac{dv}{1-vv},$$

unde, cum sit $v=\frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, erit

$$\int\frac{dv}{1-vv}=\frac{1}{2}l\frac{1+v}{1-v}=\frac{1}{2}l\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{\sqrt{(1+x^4)}-x\sqrt{2}}=\frac{1}{2}l\frac{(\sqrt{(1+x^4)}+x\sqrt{2})^2}{(1-xx)^2}$$
$$=l\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx}.$$

Deinde vero est

$$\int\frac{dv}{1+vv}=\mathrm{A\ tang.}\,v=\mathrm{A\ sin.}\frac{v}{\sqrt{(1+vv)}}=\mathrm{A\ sin.}\frac{x\sqrt{2}}{1+xx}.$$

SCHOLION

44. Quanquam autem haec quatuor exempla ad rationalitatem reducere licuit, tamen conclusio supra memorata, quod omnes formulae integrales, quae nullo modo rationales effici queant, ad aliud pertineant transcendentium genus neque per solos logarithmos et arcus circulares expediri possint, non solum manet suspecta, sed etiam falsitas eius evidenter ob oculos poni potest. Sit enim functio

$$X = \frac{a}{\sqrt{(1+xx)}} + \frac{b}{\sqrt[3]{(1+x^3)}} + \frac{c}{\sqrt[4]{(1+x^4)}};$$

tum certe formula differentialis Xdx nullo modo ad rationalitatem perduci poterit; interim tamen singulae eius partes

$$\frac{a\,dx}{\sqrt{(1+xx)}}, \quad \frac{b\,dx}{\sqrt[3]{(1+x^3)}} \quad \text{et} \quad \frac{c\,dx}{\sqrt[4]{(1+x^4)}}$$

seorsim rationales effici et integralia per logarithmos et arcus circulares exhiberi possunt. Coronidis loco hic sequens problema notatu dignum adiungamus.

PROBLEMA 14

45. *Formularum integralium*

$$\int \frac{dx}{\sqrt{(1+x^4)}} \quad et \quad \int \frac{dv}{\sqrt{(1-v^4)}}$$

valores per series investigare pro casibus, quibus ponitur tam $v=1$ *quam* $x=1$.

SOLUTIO

Cum posito $v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, ut supra fecimus, evidens sit sumto $x=0$ fore etiam $v=0$ et sumto $x=1$ fore $v=1$, ita ut hae duae quantitates x et v simul evanescant et simul unitati aequentur, hinc deducimus istam aequationem differentialem attentione dignissimam

$$\frac{1}{\sqrt{2}} \cdot \frac{dv}{\sqrt{(1-v^4)}} = \frac{dx}{\sqrt{(1+x^4)}},$$

quas ergo ambas formulas in series converti oportet; erit autem

$$\frac{1}{\sqrt{(1-v^4)}}=(1-v^4)^{-\frac{1}{2}}=1+\frac{1}{2}v^4+\frac{1\cdot 3}{2\cdot 4}v^8+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}v^{12}+\text{etc.}$$

et

$$\frac{1}{\sqrt{(1+x^4)}}=(1+x^4)^{-\frac{1}{2}}=1-\frac{1}{2}x^4+\frac{1\cdot 3}{2\cdot 4}x^8-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^{12}+\text{etc.}$$

Illa iam per dv multiplicata et integrata praebet

$$\int\frac{dv}{\sqrt{(1-v^4)}}=v+\frac{1}{2\cdot 5}v^5+\frac{1\cdot 3}{2\cdot 4\cdot 9}v^9+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}v^{13}+\text{etc.},$$

unde posito $v=1$ valor huius integralis erit

$$1+\frac{1}{2\cdot 5}+\frac{1\cdot 3}{2\cdot 4\cdot 9}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 17}+\text{etc.},$$

quam seriem littera A indicemus. Simili modo altera series in dx ducta et integrata producit

$$\int\frac{dx}{\sqrt{(1+x^4)}}=x-\frac{1}{2\cdot 5}x^5+\frac{1\cdot 3}{2\cdot 4\cdot 9}x^9-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}x^{13}+\text{etc.},$$

cuius valor facto $x=1$ erit

$$1-\frac{1}{2\cdot 5}+\frac{1\cdot 3}{2\cdot 4\cdot 9}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 13}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 17}-\text{etc.},$$

quem littera B designemus, ita ut sit $B=\frac{A}{\sqrt{2}}$ sive $A=B\sqrt{2}$, unde patet priorem seriem se habere ad posteriorem ut $\sqrt{2}:1$.

SCHOLION

46. Valor formulae integralis $\int\frac{dv}{\sqrt{(1-v^4)}}$ etiam hoc modo per seriem investigari potest. Cum sit

$$\frac{1}{\sqrt{(1-v^4)}}=\frac{(1+vv)^{-\frac{1}{2}}}{\sqrt{(1-vv)}}$$

et

$$(1+vv)^{-\frac{1}{2}}=1-\frac{1}{2}vv+\frac{1\cdot 3}{2\cdot 4}v^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}v^6+\text{etc.},$$

notetur esse $\int \frac{dv}{\sqrt{(1-vv)}} = \frac{\pi}{2}$. Deinde pro integratione reliquorum terminorum ponatur

$$\int \frac{v^{n+2}dv}{\sqrt{(1-vv)}} = Av^{n+1}\sqrt{(1-vv)} + B\int \frac{v^n dv}{\sqrt{(1-vv)}},$$

quae aequatio differentiata dat

$$\frac{v^{n+2}}{\sqrt{(1-vv)}} = (n+1)Av^n\sqrt{(1-vv)} - \frac{Av^{n+2}}{\sqrt{(1-vv)}} + \frac{Bv^n}{\sqrt{(1-vv)}},$$

unde per $\sqrt{(1-vv)}$ multiplicando prodit

$$v^{n+2} = (n+1)Av^n - (n+1)Av^{n+2} - Av^{n+2} + Bv^n.$$

Hinc termini, in quibus inest v^{n+2}, inter se aequati praebent $1 = -(n+2)A$ ideoque $A = -\frac{1}{n+2}$, termini vero v^n continentes praebent $0 = (n+1)A + B$, unde fit $B = \frac{n+1}{n+2}$, ita ut in genere sit

$$\int \frac{v^{n+2}dv}{\sqrt{(1-vv)}} = -\frac{1}{n+2}v^{n+1}\sqrt{(1-vv)} + \frac{n+1}{n+2}\int \frac{v^n dv}{\sqrt{(1-vv)}},$$

quod integrale, uti requiritur, evanescit posito $v = 0$. Ponatur nunc $v = 1$ eritque

$$\int \frac{v^{n+2}dv}{\sqrt{(1-vv)}} = \frac{n+1}{n+2}\int \frac{v^n dv}{\sqrt{(1-vv)}};$$

hinc ergo pro n scribendo successive valores 0, 2, 4, 6, 8 etc. erit

$$\text{I.}\quad \int \frac{vv\,dv}{\sqrt{(1-vv)}} = \frac{1}{2}\cdot\frac{\pi}{2},$$

$$\text{II.}\quad \int \frac{v^4 dv}{\sqrt{(1-vv)}} = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

$$\text{III.}\quad \int \frac{v^6 dv}{\sqrt{(1-vv)}} = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

etc.,

quibus valoribus adhibitis erit casu $v=1$

$$\int\frac{dv}{\sqrt{(1-v^4)}}=\frac{\pi}{2}-\frac{1^2}{2^2}\cdot\frac{\pi}{2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{\pi}{2}-\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi}{2}+\text{etc.}$$
$$=\frac{\pi}{2}\left(1-\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}-\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}-\text{etc.}\right),$$

ita ut sit ex problemate praecedente

$$1-\frac{1}{2\cdot 5}+\frac{1\cdot 3}{2\cdot 4\cdot 9}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 9\cdot 13}+\text{etc.}=\frac{\pi}{2}\left(1-\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}-\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\text{etc.}\right),$$

unde fit

$$\frac{\pi}{2}=\frac{1-\frac{1}{2\cdot 5}+\frac{1\cdot 3}{2\cdot 4\cdot 9}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 9\cdot 13}+\text{etc.}}{1-\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}-\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\text{etc.}}.$$

NOVA METHODUS INTEGRANDI FORMULAS DIFFERENTIALES RATIONALES SINE SUBSIDIO QUANTITATUM IMAGINARIARUM

Commentatio 572 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1781: I, 1784, p. 3—47

THEOREMA 1

1. *Si fuerit*

$$xx - 2x\cos.\omega + 1 = 0,$$

tum omnes potestates ipsius x reduci poterunt ad formam simplicem

$$\alpha x + \beta.$$

DEMONSTRATIO

Cum sit $xx - 2x\cos.\omega + 1 = 0$, erit $x^{\lambda+2} = 2x^{\lambda+1}\cos.\omega - x^{\lambda}$; unde si potestates x^{λ} et $x^{\lambda+1}$ ad formam praescriptam $\alpha x + \beta$ redigi queant, tum etiam potestas $x^{\lambda+2}$ per eandem formam exprimi poterit. Incipiamus igitur a potestatibus infimis, quas ita exhibeamus

$$x = \frac{x\sin.\omega}{\sin.\omega} \quad \text{et} \quad xx = \frac{2x\sin.\omega\cos.\omega - \sin.\omega}{\sin.\omega} = \frac{x\sin.2\omega - \sin.\omega}{\sin.\omega}.$$

His igitur constitutis, cum sit

$$2\cos.\omega\sin.\lambda\omega = \sin.(\lambda+1)\omega + \sin.(\lambda-1)\omega,$$

ex his duabus formulis facile eliciemus sequentes

$$x^3 = 2xx\cos.\omega - x = \frac{x\sin.3\omega - \sin.2\omega}{\sin.\omega},$$

$$x^4 = 2x^3\cos.\omega - xx = \frac{x\sin.4\omega - \sin.3\omega}{\sin.\omega}$$

hocque modo quousque libuerit progredi licet; atque hinc in genere concludimus fore

$$x^n = \frac{x\sin.n\omega - \sin.(n-1)\omega}{\sin.\omega},$$

quae ergo expressio formam habet $\alpha x + \beta$.

COROLLARIUM 1

2. Cum sit

$$\sin.(n-1)\omega = \sin.n\omega\cos.\omega - \cos.n\omega\sin.\omega,$$

hoc valore substituto fiet

$$x^n = \frac{x-\cos.\omega}{\sin.\omega}\sin.n\omega + \cos.n\omega;$$

quamobrem si fuerit $xx - 2x\cos.\omega + 1 = 0$, pro omnibus potestatibus ipsius x habebimus hanc reductionem

$$x^n = \frac{x-\cos.\omega}{\sin.\omega}\sin.n\omega + \cos.n\omega,$$

qua forma deinceps potissimum utemur.

COROLLARIUM 2

3. Hinc igitur erit

$$x^{k+n} = \frac{x-\cos.\omega}{\sin.\omega}\sin.(k+n)\omega + \cos.(k+n)\omega$$

et

$$x^{k-n} = \frac{x-\cos.\omega}{\sin.\omega}\sin.(k-n)\omega + \cos.(k-n)\omega,$$

quare his formulis addendis ob

$$\sin.(k+n)\omega + \sin.(k-n)\omega = 2\sin.k\omega\cos.n\omega$$

et

$$\cos.(k+n)\omega + \cos.(k-n)\omega = 2\cos.k\omega\cos.n\omega$$

fiet

$$x^{k+n} + x^{k-n} = \frac{2(x-\cos.\omega)}{\sin.\omega}\sin.k\omega\cos.n\omega + 2\cos.k\omega\cos.n\omega$$

sive

$$x^{k+n} + x^{k-n} = 2\cos.n\omega\left(\frac{x-\cos.\omega}{\sin.\omega}\sin.k\omega + \cos.k\omega\right).$$

COROLLARIUM 3

4. Sin autem potestatem posteriorem a priore subtrahamus, ob

$$\sin.(k+n)\omega - \sin.(k-n)\omega = 2\cos.k\omega\sin.n\omega$$

et

$$\cos.(k+n)\omega - \cos.(k-n)\omega = -2\sin.k\omega\sin.n\omega$$

habebimus

$$x^{k+n} - x^{k-n} = \frac{2(x-\cos.\omega)}{\sin.\omega}\cos.k\omega\sin.n\omega - 2\sin.k\omega\sin.n\omega;$$

hoc est

$$x^{k+n} - x^{k-n} = 2\sin.n\omega\left(\frac{x-\cos.\omega}{\sin.\omega}\cos.k\omega - \sin.k\omega\right).$$

COROLLARIUM 4

5. Etiamsi nostra demonstratio tantum ad potestates integras ipsius x perduxit, tamen ex indole harum formarum facile intelligitur eas etiam pro exponentibus fractis vel adeo irrationalibus locum habere, quandoquidem in ipsis his formulis nihil inest, quod tantum ad valores integros exponentis n restringatur; tum vero etiam nihil impedit, quominus exponenti n valores negativi tribuantur. Si enim verbi gratia sumamus $n=\frac{1}{2}$, per formulam Corollarii 1 esse debet

$$\sqrt{x} = \frac{x-\cos.\omega}{\sin.\omega}\sin.\frac{1}{2}\omega + \cos.\frac{1}{2}\omega,$$

unde sumtis quadratis ob

$$(x-\cos.\omega)^2 = xx - 2x\cos.\omega + \cos.\omega^2 = -\sin.\omega^2$$

habebimus

$$x = -\sin.\frac{1}{2}\omega^2 + \frac{2(x-\cos.\omega)}{\sin.\omega}\sin.\frac{1}{2}\omega\cos.\frac{1}{2}\omega + \cos.\frac{1}{2}\omega^2,$$

quae forma ob

$$2\sin.\frac{1}{2}\omega\cos.\frac{1}{2}\omega = \sin.\omega \quad \text{et} \quad \cos.\frac{1}{2}\omega^2 - \sin.\frac{1}{2}\omega^2 = \cos.\omega$$

abit in $x = x$, hoc est aequationem identicam.

SCHOLION

6. Formulae, quas hic sumus adepti, egregie conveniunt cum iis, quas calculus imaginariorum suppeditat. Cum enim aequatio $xx - 2x\cos.\omega + 1 = 0$ contineat has radices

$$x = \cos.\omega \pm \sin.\omega\sqrt{-1},$$

erit, uti in analysi est ostensum,

$$x^n = \cos.n\omega \pm \sin.n\omega\sqrt{-1};$$

quare cum sit

$$x^n - \cos.n\omega = \pm\sin.n\omega\sqrt{-1} \quad \text{et} \quad x - \cos.\omega = \pm\sin.\omega\sqrt{-1},$$

illa forma per hanc divisa dabit

$$\frac{x^n - \cos.n\omega}{x - \cos.\omega} = \frac{\sin.n\omega}{\sin.\omega},$$

unde sequitur fore

$$x^n - \cos.n\omega = \frac{x-\cos.\omega}{\sin.\omega}\sin.n\omega,$$

prorsus uti in Corollario 1 invenimus.

Ceterum nostrum theorema generalius proponi et ad aequationem

$$xx - 2ax\cos.\omega + aa = 0$$

extendi potuisset; tum enim prodiisset

$$x^n = \frac{a^{n-1}x\sin.n\omega - a^n\sin.(n-1)\omega}{\sin.\omega},$$

deinde etiam

$$x^n = \frac{a^{n-1}(x - a\cos.\omega)}{\sin.\omega}\sin.n\omega + a^n\cos.n\omega,$$

quae formulae a prioribus non discrepant, nisi quod hic littera a homogeneitatem dimensionum expleat. Hae scilicet formulae ex illis immediate sequentur, si ibi loco x scribatur $\frac{x}{a}$; sed brevitati et concinnitati consulentes eiusmodi tantum casus evolvemus, in quibus pro a commode unitatem scribere liceat.

THEOREMA 2

7. *Si fuerit*

$$xx - 2x\cos.\omega + 1 = 0,$$

omnes functiones rationales integrae, quaecunque potestates ipsius x in iis occurrant, semper reduci possunt ad hanc formam simplicem

$$\alpha x + \beta.$$

DEMONSTRATIO

Si functio proposita iam penitus fuerit evoluta, ita ut nullos factores complectatur, tum ea ope reductionis

$$x^n = \frac{x - \cos.\omega}{\sin.\omega}\sin.n\omega + \cos.n\omega$$

sponte redigitur ad talem formam $\frac{F(x-\cos.\omega)}{\sin.\omega} + G$. Verum si functio proposita duobus constet factoribus, veluti Pp, ac per istam reductionem prodierit

$$P = \frac{F(x-\cos.\omega)}{\sin.\omega} + G \quad \text{et} \quad p = \frac{f(x-\cos.\omega)}{\sin.\omega} + g,$$

tum facta multiplicatione ob $(x - \cos.\omega)^2 = -\sin.\omega^2$ colligitur fore

$$Pp = -Ff + Gg + \frac{(Fg + fG)(x - \cos.\omega)}{\sin.\omega},$$

quod ergo productum eiusdem est formae; unde simul patet, quotcunque eiusmodi dentur factores, eorum productum semper ad eandem formam reduci posse.

COROLLARIUM 1

8. Quodsi hoc modo prodierit

$$P = \frac{F(x - \cos.\omega)}{\sin.\omega} + G,$$

tum erit

$$P(x - \cos.\omega) = -F\sin.\omega + G(x - \cos.\omega),$$

quae expressio ideo est notatu digna, quod in sequentibus integrationibus ubique occurret.

COROLLARIUM 2

9. Si functio P factorem habuerit $xx - 2x\cos.\omega + 1$, tum posito, uti assumsimus,

$$xx - 2x\cos.\omega + 1 = 0,$$

valor ipsius P etiam evanescere debet. Hoc ergo casu formula $\frac{F(x - \cos.\omega)}{\sin.\omega} + G$ fiet $= 0$, id quod ob x quantitatem indefinitam aliter evenire nequit, nisi fuerit et $F = 0$ et $G = 0$. Atque hinc vicissim, si facta reductione prodeat $P = 0$, hoc certum erit signum ipsam functionem involvere factorem

$$xx - 2x\cos.\omega + 1.$$

THEOREMA 3

10. *Si fuerit*

$$xx - 2x\cos.\omega + 1 = 0,$$

tum etiam omnes functiones fractae rationales semper ad formam simplicem

$$\alpha x + \beta$$

reduci possunt.

DEMONSTRATIO

Sit enim proposita functio quaecunque fracta $\frac{P}{p}$ atque adhibita nostra reductione prodierit

$$P = \frac{F(x - \cos.\omega)}{\sin.\omega} + G \quad \text{et} \quad p = \frac{f(x - \cos.\omega)}{\sin.\omega} + g,$$

ita ut pervenerimus ad hanc fractionem

$$\frac{P}{p} = \frac{\frac{F(x-\cos.\omega)}{\sin.\omega} + G}{\frac{f(x-\cos.\omega)}{\sin.\omega} + g}.$$

Iam ut ipsam litteram x ex denominatore expellamus, multiplicemus tam numeratorem quam denominatorem per formulam $\frac{f(x-\cos.\omega)}{\sin.\omega} - g$; sic enim ob

$$(x-\cos.\omega)^2 = -\sin.\omega^2$$

pro denominatore reperiemus $-ff-gg$, at vero pro numeratore

$$-Ff + \frac{(fG-Fg)(x-\cos.\omega)}{\sin.\omega} - Gg,$$

unde mutatis signis forma nostrae fractionis erit

$$\frac{P}{p} = \frac{\frac{(Fg-fG)(x-\cos.\omega)}{\sin.\omega} + Ff + Gg}{ff+gg}$$

sive concinnius

$$\frac{P}{p} = \frac{Fg-fG}{ff+gg}\cdot\frac{x-\cos.\omega}{\sin.\omega} + \frac{Ff+Gg}{ff+gg}.$$

PROBLEMA

11. *Proposita formula differentiali rationali quacunque, eam in suas fractiones partiales resolvere ac deinceps eius integrale investigare.*

SOLUTIO

Repraesentetur formula differentialis sub hac specie $\frac{P}{Q}\cdot\frac{dx}{x} = \frac{Pdx}{Qx}$, ita tamen, ut $\frac{P}{x}$ maneat functio integra, ne x sit factor denominatoris. Ante omnia quaerantur igitur ipsius Q omnes factores tam simplices quam duplices reales; et quia simplices nulla laborant difficultate, hic tantum duplices sum contemplaturus, quorum forma sit $xx-2x\cos.\omega+1$, ita ut posito $xx-2x\cos.\omega+1=0$ quantitas Q simul in nihilum abeat; ex qua conditione omnes valores anguli ω elici poterunt, ita ut hoc modo omnes factores

denominatoris Q obtineantur. Nunc igitur fractionem $\frac{P}{Qx}$ in totidem fractiones partiales resolvi oportet, quot inventi fuerint factores formae

$$xx - 2x \cos. \omega + 1.$$

Sit igitur in genere fractio partialis ex isto factore nata

$$= \frac{\alpha x + \beta}{xx - 2x \cos. \omega + 1},$$

quandoquidem novimus eius numeratorem talem formam $\alpha x + \beta$ habere debere; pro reliquis autem fractionibus partialibus omnibus scribamus litteram R, ita ut esse debeat

$$\frac{P}{Qx} = \frac{\alpha x + \beta}{xx - 2x \cos. \omega + 1} + R,$$

et multiplicando per $xx - 2x \cos. \omega + 1$ habebimus

$$\frac{P(xx - 2x \cos. \omega + 1)}{Qx} = \alpha x + \beta + R(xx - 2x \cos. \omega + 1).$$

Quodsi ergo iam faciamus $xx - 2x \cos. \omega + 1 = 0$, erit

$$\alpha x + \beta = \frac{P(xx - 2x \cos. \omega + 1)}{Qx} = \frac{P}{x} \cdot \frac{xx - 2x \cos. \omega + 1}{Q},$$

ubi in priore factore $\frac{P}{x}$ ista substitutio nullam habet difficultatem; verum in altera fractione $\frac{xx - 2x \cos. \omega + 1}{Q}$, quia posito $xx - 2x \cos. \omega + 1 = 0$ non solum numerator, sed etiam denominator Q evanescit, secundum praecepta cognita utriusque loco eius differentiale scribamus, siquidem hoc casu fieri debet

$$\frac{xx - 2x \cos. \omega + 1}{Q} = \frac{2dx(x - \cos. \omega)}{dQ},$$

sicque obtinebitur numerator quaesitus

$$\alpha x + \beta = \frac{2Pdx(x - \cos. \omega)}{xdQ}.$$

Ponamus igitur per hanc substitutionem fieri

$$P = \frac{F(x - \cos. \omega)}{\sin. \omega} + G \quad \text{et} \quad \frac{xdQ}{dx} = \frac{f(x - \cos. \omega)}{\sin. \omega} + g,$$

ita ut sit

$$\frac{Pdx}{xdQ} = \frac{\frac{F(x-\cos.\omega)}{\sin.\omega} + G}{\frac{f(x-\cos.\omega)}{\sin.\omega} + g},$$

quae forma per theorema tertium reducitur ad hanc

$$\frac{Pdx}{xdQ} = \frac{Fg-fG}{ff+gg} \cdot \frac{x-\cos.\omega}{\sin.\omega} + \frac{Ff+Gg}{ff+gg},$$

quae ergo insuper per $2(x-\cos.\omega)$ multiplicata ob $(x-\cos.\omega)^2 = -\sin.\omega^2$ praebet numeratorem quaesitum

$$\alpha x + \beta = \frac{2(Ff+Gg)(x-\cos.\omega)}{ff+gg} + \frac{2(fG-Fg)}{ff+gg}\sin.\omega.$$

Multiplicetur igitur ista forma per $\frac{dx}{xx-2x\cos.\omega+1}$ atque obtinebitur pars integralis ex hac fractione partiali oriunda

$$\frac{2(Ff+Gg)}{ff+gg}\int\frac{dx(x-\cos.\omega)}{xx-2x\cos.\omega+1} + \frac{2(fG-Fg)}{ff+gg}\sin.\omega\int\frac{dx}{xx-2x\cos.\omega+1}.$$

Hic igitur pro priore parte manifesto est

$$\int\frac{dx(x-\cos.\omega)}{xx-2x\cos.\omega+1} = l\sqrt{(xx-2x\cos.\omega+1)},$$

quod integrale iam ita est sumtum, ut evanescat posito $x=0$; pro altero autem membro facile reperitur

$$\int\frac{dx\sin.\omega}{1-2x\cos.\omega+xx} = \text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

quod itidem evanescit posito $x=0$, quocirca pars integralis ex denominatoris Q factore $xx-2x\cos.\omega+1$ orta erit

$$\frac{2(Ff+Gg)}{ff+gg}l\sqrt{(xx-2x\cos.\omega+1)} + \frac{2(fG-Fg)}{ff+gg}\text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega}.$$

COROLLARIUM 1

12. Duo autem casus hic singularem evolutionem postulant, alter, quo $\omega = 0$, alter vero, quo $\omega = 180^0$; priore enim casu denominator $xx - 2x \cos. \omega + 1$ abit in $(x-1)^2$, posteriore vero in $(x+1)^2$. Cum autem hinc plus concludere non liceat quam vel $1 - x$ vel $1 + x$ esse factorem denominatoris, his casibus pars integralis in genere inventa tantum ad semissem redigi debet, quemadmodum in principiis calculi integralis fusius est ostensum.[1]) Ceterum his casibus posterior pars a circulo pendens semper evanescet.

COROLLARIUM 2

13. Praeter hos autem binos casus portio integralis ex formula $xx - 2x \cos. \omega + 1$ oriunda semper constabit duabus partibus, altera logarithmica, altera circulari, nisi forte fuerit vel $Ff + Gg = 0$ vel $fG - gF = 0$. Priore enim casu haec portio tantum arcum circularem involvet, posteriore vero tantum logarithmum.

SCHOLION

14. Quoniam assumsimus denominatoris Q factorem esse $xx - 2x \cos. \omega + 1$, alias denominatoris formas hic non contemplabimur, nisi quarum omnes factores tali formula exprimi queant. Tales autem formulae simpliciores occurrunt tres sequentes:

$$Q = 1 + x^{2k}, \quad Q = 1 - x^{2k}, \quad Q = 1 + 2x^k \cos. \eta + x^{2k},$$

ubi quidem in prioribus potestati ipsius x exponentem parem tribuimus, quoniam casus, quibus esset impar, facile ad hanc formam reduci possunt. Si enim denominator esset $1 \pm x^i$ denotante i numerum imparem, tantum loco x scribamus y^2 prodibitque talis forma $1 \pm y^{2i}$; at tali substitutione natura formulae differentialis neutiquam mutatur. Hos ergo tres casus in sequentibus tribus problematibus particularibus omni cura percurramus, quo magis praestantia istius novae methodi prae aliis, quae adhuc in usu fuerunt, eluceat.

1) Vide L. Euleri *Institutionum calculi integralis* vol. I, § 77, Petropoli 1768; *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 41. Vide etiam Euleri Commentationem 462 (indicis Enestroemiani): *De valore formulae integralis* $\int \frac{z^{m-1} \pm z^{n-m-1}}{1 \pm z^n} dz$ *casu, quo post integrationem ponitur* $z = 1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 3; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 358, imprimis p. 369. Cf. porro eiusdem voluminis 17 Commentationes 60 et 162, imprimis p. 59, 126, 134. A. G.

PROBLEMA PARTICULARE 1

15. *Si fuerit*

$$Q = 1 + x^{2k},$$

investigare integrale huius formulae differentialis

$$\frac{P\,dx}{(1+x^{2k})\,x},$$

ubi quidem $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$, *ne scilicet ista fractio evadat spuria.*

SOLUTIO

Cum sit $Q = 1 + x^{2k}$, sit eius factor trinomialis quicunque

$$= xx - 2x\cos.\omega + 1,$$

ita ut numerus talium factorum sit $= k$; quare cum posito

$$xx - 2x\cos.\omega + 1 = 0$$

etiam ipsa formula $1 + x^{2k}$ evanescere debeat, facta substitutione debita secundum Theorema 2 fiet

$$Q = 1 + \frac{(x - \cos.\omega)}{\sin.\omega}\sin.2k\omega + \cos.2k\omega;$$

qui valor cum debeat evanescere, erit tam $\sin.2k\omega = 0$ quam $1 + \cos.2k\omega = 0$. Conditio ergo posterior praebet $\cos.2k\omega = -1$; unde intelligitur angulum $2k\omega$ esse debere vel π vel 3π vel 5π vel in genere $(2i-1)\pi$ denotante $2i-1$ numerum imparem quemcunque. Valores igitur anguli ω erunt sequentes:

$$1.\quad \omega = \frac{\pi}{2k}, \qquad 2.\quad \omega = \frac{3\pi}{2k}, \qquad 3.\quad \omega = \frac{5\pi}{2k}$$

et generatim

$$\omega = \frac{(2i-1)\pi}{2k};$$

quorum numerus cum esse debeat $= k$, ultimus valor erit $\omega = \frac{(2k-1)\pi}{2k}$; singulis autem istis valoribus simul prior conditio adimpletur, qua esse debet

$\sin. 2k\omega = 0$. Quodsi iam pro ω unusquisque horum valorum accipiatur atque ponatur

$$xx - 2x \cos. \omega + 1 = 0,$$

quicunque fuerit numerator P, sumamus facta hac substitutione fieri

$$P = \frac{F(x - \cos. \omega)}{\sin. \omega} + G;$$

tum vero erit $\frac{xdQ}{dx} = 2kx^{2k}$, unde, cum nostro casu fieri debeat $Q = 0$, erit utique $x^{2k} = -1$ sicque fiet $\frac{xdQ}{dx} = -2k$. Cum igitur haec formula in genere posita sit $\frac{f(x - \cos. \omega)}{\sin. \omega} + g$, erit nunc $f = 0$ et $g = -2k$, quo invento secundum praecepta ante tradita pars integralis ex hoc factore denominatoris

$$xx - 2x \cos. \omega + 1$$

oriunda erit

$$-\frac{G}{k} l\sqrt{(xx - 2x \cos. \omega + 1)} + \frac{F}{k} \text{A tang.} \frac{x \sin. \omega}{1 - x \cos. \omega};$$

consequenter si ex singulis valoribus anguli ω istae partes integralis formentur et in unam summam colligantur, impetrabitur totum integrale formulae differentialis propositae; et quia hoc casu nunquam fieri potest vel $\omega = 0$ vel $\omega = \pi$, cautione supra indicata non erit opus.

COROLLARIUM 1

16. Quodsi numerator P fuerit potestas simplex ipsius x, puta x^m existente $m > 0$, at $m < 2k$, ut formula integranda sit

$$\int \frac{x^{m-1}dx}{1 + x^{2k}},$$

posito $xx - 2x \cos. \omega + 1 = 0$ erit formula

$$P = x^m = \frac{(x - \cos. \omega)}{\sin. \omega} \sin. m\omega + \cos. m\omega$$

ideoque

$$F = \sin. m\omega \quad \text{et} \quad G = \cos. m\omega,$$

unde quaelibet portio integralis induet hanc formam

$$-\frac{\cos. m\omega}{k} l\sqrt{(xx - 2x \cos. \omega + 1)} + \frac{\sin. m\omega}{k} \text{A tang.} \frac{x \sin. \omega}{1 - x \cos. \omega},$$

et aggregatum omnium harum partium, siquidem loco ω successive singuli eius valores substituantur, dabit totum integrale formulae huius propositae ita sumtum, ut evanescat posito $x = 0$.

COROLLARIUM 2

17. Si numerator P ex pluribus huiusmodi terminis constet, ut sit $P = ax^\alpha + bx^\beta + cx^\gamma +$ etc., integratio maiore difficultate non laborat; erit enim

$$F = a \sin. \alpha\omega + b \sin. \beta\omega + c \sin. \gamma\omega + \text{etc.}$$

et

$$G = a \cos. \alpha\omega + b \cos. \beta\omega + c \cos. \gamma\omega + \text{etc.}$$

hincque totum integrale facile expedietur.

SCHOLION

18. Hic autem occurrit casus imprimis memorabilis, quo sumitur $P = x^{k-n} + x^{k+n}$, quem in sequente Problemate speciali seorsim evolvamus.

PROBLEMA SPECIALE

19. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$$

eius totum integrale evolvere.

SOLUTIO

Cum hic sit $P = x^{k-n} + x^{k+n}$, si statuamus

$$xx - 2x \cos. \omega + 1 = 0,$$

fiet

$$P = \frac{x - \cos. \omega}{\sin. \omega} (\sin. (k-n)\omega + \sin. (k+n)\omega) + \cos. (k-n)\omega + \cos. (k+n)\omega,$$

unde sponte se produnt litterae F et G; cum autem in genere sit

$$\sin. p + \sin. q = 2 \sin. \frac{p+q}{2} \cos. \frac{p-q}{2}$$

et

$$\cos. p + \cos. q = 2 \cos. \frac{p+q}{2} \cos. \frac{p-q}{2},$$

facta hac reductione reperietur

$$F = 2\sin. k\omega \cos. n\omega \quad \text{et} \quad G = 2\cos. k\omega \cos. n\omega.$$

Cum autem in genere sit $\omega = \frac{(2i-1)\pi}{2k}$, erit $\sin. k\omega = \sin. \frac{(2i-1)\pi}{2}$, cuius valor est vel $+1$ vel -1; utrumvis autem locum habeat, semper erit $\cos. k\omega = 0$, ita ut sit

$$F = 2\sin. \frac{(2i-1)\pi}{2} \cos. n\frac{(2i-1)\pi}{2k} \quad \text{et} \quad G = 0;$$

quibus valoribus inventis pars integralis ex hoc factore generali oriunda erit

$$\frac{2}{k}\sin. \frac{(2i-1)\pi}{2} \cos. \frac{(2i-1)n\pi}{2k} \,\text{A tang.}\, \frac{x\sin. \frac{(2i-1)\pi}{2k}}{1 - x\cos. \frac{(2i-1)\pi}{2k}}.$$

Hinc ergo, si loco i successive scribamus valores 1, 2, 3, 4 etc. usque ad k, totum integrale quaesitum sequenti forma exprimetur:

$$\frac{2}{k}\cos. \frac{n\pi}{2k} \,\text{A tang.}\, \frac{x\sin. \frac{\pi}{2k}}{1 - x\cos. \frac{\pi}{2k}} - \frac{2}{k}\cos. \frac{3n\pi}{2k} \,\text{A tang.}\, \frac{x\sin. \frac{3\pi}{2k}}{1 - x\cos. \frac{3\pi}{2k}}$$

$$+ \frac{2}{k}\cos. \frac{5n\pi}{2k} \,\text{A tang.}\, \frac{x\sin. \frac{5\pi}{2k}}{1 - x\cos. \frac{5\pi}{2k}} - \frac{2}{k}\cos. \frac{7n\pi}{2k} \,\text{A tang.}\, \frac{x\sin. \frac{7\pi}{2k}}{1 - x\cos. \frac{7\pi}{2k}}$$

$$\vdots$$

$$+ \frac{2}{k}\sin. \frac{(2k-1)\pi}{2} \cos. \frac{(2k-1)n\pi}{2k} \,\text{A tang.}\, \frac{x\sin. \frac{(2k-1)\pi}{2k}}{1 - x\cos. \frac{(2k-1)\pi}{2k}},$$

ubi imprimis notatu dignum usu venit, ut omnes partes logarithmicae se mutuo destruxerint.

COROLLARIUM 1

20. Quodsi ergo sumamus $n = 0$, ita ut formula integranda sit

$$\int \frac{2x^k}{1 + x^{2k}} \cdot \frac{dx}{x},$$

eius integrale hoc modo exprimetur:

$$\frac{2}{k}\,\text{A tang.}\,\frac{x\sin.\frac{\pi}{2k}}{1-x\cos.\frac{\pi}{2k}}-\frac{2}{k}\,\text{A tang.}\,\frac{x\sin.\frac{3\pi}{2k}}{1-x\cos.\frac{3\pi}{2k}}+\frac{2}{k}\,\text{A tang.}\,\frac{x\sin.\frac{5\pi}{2k}}{1-x\cos.\frac{5\pi}{2k}}-\cdots$$

$$+\frac{2}{k}\sin.\frac{(2k-1)\pi}{2}\,\text{A tang.}\,\frac{x\sin.\frac{(2k-1)\pi}{2k}}{1-x\cos.\frac{(2k-1)\pi}{2k}}.$$

At posito $x^k=z$ ob $\frac{dx}{x}=\frac{dz}{kz}$ formula integralis induet hanc formam $\int\frac{2dz}{k(1+zz)}$, cuius integrale manifesto est

$$\frac{2}{k}\,\text{A tang.}\,z=\frac{2}{k}\,\text{A tang.}\,x^k,$$

unde sequitur fore

$$\text{A tang.}\,x^k$$

$$=\text{A tang.}\,\frac{x\sin.\frac{\pi}{2k}}{1-x\cos.\frac{\pi}{2k}}-\text{A tang.}\,\frac{x\sin.\frac{3\pi}{2k}}{1-x\cos.\frac{3\pi}{2k}}+\text{A tang.}\,\frac{x\sin.\frac{5\pi}{2k}}{1-x\cos.\frac{5\pi}{2k}}-\cdots$$

$$+\sin\frac{(2k-1)\pi}{2}\,\text{A tang.}\,\frac{x\sin.\frac{(2k-1)\pi}{2k}}{1-x\cos.\frac{(2k-1)\pi}{2k}},$$

quod sane est theorema maxima attentione dignum.

COROLLARIUM 2

21. Ad hoc theorema illustrandum sumamus $k=1$ et ob $\sin.\frac{\pi}{2}=1$ et $\cos.\frac{\pi}{2}=0$ prodit manifesto $\text{A tang.}\,x=\text{A tang.}\,x$.

At sumto $k=2$ ob

$$\sin.\frac{\pi}{4}=\frac{1}{\sqrt{2}},\quad \cos.\frac{\pi}{4}=\frac{1}{\sqrt{2}},\quad \sin.\frac{3\pi}{4}=\frac{1}{\sqrt{2}}\quad\text{et}\quad \cos.\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}$$

fiet

$$\text{A tang.}\,xx=\text{A tang.}\,\frac{x}{\sqrt{2}-x}-\text{A tang.}\,\frac{x}{\sqrt{2}+x}.$$

Cum autem in genere sit

$$\text{A tang.}\,p-\text{A tang.}\,q=\text{A tang.}\,\frac{p-q}{1+pq},$$

hoc casu erit

$$p = \frac{x}{\sqrt{2} - x} \quad \text{et} \quad q = \frac{x}{\sqrt{2} + x}$$

ideoque

$$p - q = \frac{2xx}{2 - xx} \quad \text{et} \quad 1 + pq = \frac{2}{2 - xx},$$

unde manifesto prodit A tang. xx = A tang. xx.

Sumamus porro $k = 3$ et ob

$$\sin.\frac{\pi}{6} = \frac{1}{2}, \quad \cos.\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \sin.\frac{3\pi}{6} = 1, \quad \cos.\frac{3\pi}{6} = 0,$$

$$\sin.\frac{5\pi}{6} = \frac{1}{2} \quad \text{et} \quad \cos.\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

reperietur

$$\text{A tang.}\, x^3 = \text{A tang.}\, \frac{x}{2 - x\sqrt{3}} - \text{A tang.}\, x + \text{A tang.}\, \frac{x}{2 + x\sqrt{3}},$$

ubi per reductionem superiorem arcus primus et tertius iunctim sumti ob

$$p = \frac{x}{2 - x\sqrt{3}} \quad \text{et} \quad q = \frac{-x}{2 + x\sqrt{3}}$$

praebent A tang. $\frac{x}{1 - xx}$; a quo si subtrahatur A tang. x, remanebit A tang. x^3.

SCHOLION

22. Ceterum veritas huius theorematis in genere commodissime sumendis differentialibus ostendi potest. Cum enim sit

$$d.\,\text{A tang.}\, x^k = \frac{kx^{k-1}dx}{1 + x^{2k}} \quad \text{et} \quad d.\,\text{A tang.}\, \frac{x \sin.\omega}{1 - x\cos.\omega} = \frac{dx \sin.\omega}{1 - 2x\cos.\omega + xx},$$

si loco ω valores debiti successive substituantur et per dx dividatur, resultabit sequens aequatio

$$\frac{kx^{k-1}}{1 + x^{2k}} = \frac{\sin.\frac{\pi}{2k}}{1 - 2x\cos.\frac{\pi}{2k} + xx} - \frac{\sin.\frac{3\pi}{2k}}{1 - 2x\cos.\frac{3\pi}{2k} + xx} + \cdots \pm \frac{\sin.\frac{(2k-1)\pi}{2k}}{1 - 2x\cos.\frac{(2k-1)\pi}{2k} + xx},$$

quae sunt eae ipsae fractiones partiales, in quas functio fracta $\frac{kx^{k-1}}{1 + x^{2k}}$

resolvitur. Ceterum cum in hac integratione omnes logarithmi excesserint, duplex quaestio circa integrale inventum institui potest, altera, qua quaeritur eius valor casu $x = \infty$, altera vero casu, quo sumitur $x = 1$.

QUAESTIO PRIOR

23. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + x^{2k}} \cdot \frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponitur $x = \infty$.

SOLUTIO

Cum quilibet arcus in expressione integralis inventi § 19 in genere sit huiusmodi A tang. $\frac{x \sin. \omega}{1 - x \cos. \omega}$, si statuatur $x = \infty$, is hanc induet formam A tang. $(-\operatorname{tang}. \omega)$. Quia autem $-\operatorname{tang}. \omega = +\operatorname{tang}. (\pi - \omega)$, iste arcus fiet $= \pi - \omega$; quare si loco ω successive valores debitos substituamus, integrale quaesitum sequenti serie exprimetur

$$\begin{aligned} &\frac{2}{k}\left(\pi - \frac{\pi}{2k}\right) \cos. \frac{n\pi}{2k} - \frac{2}{k}\left(\pi - \frac{3\pi}{2k}\right) \cos. \frac{3n\pi}{2k} + \frac{2}{k}\left(\pi - \frac{5\pi}{2k}\right) \cos. \frac{5n\pi}{2k} \\ &- \frac{2}{k}\left(\pi - \frac{7\pi}{2k}\right) \cos. \frac{7n\pi}{2k} + \cdots \pm \frac{2}{k}\left(\pi - \frac{(2k-1)\pi}{2k}\right) \cos. \frac{(2k-1)n\pi}{2k}, \end{aligned}$$

cuius ultimum membrum habebit signum $+$, quoties fuerit $2k - 1$ numerus formae $4\alpha + 1$ sive $k = 2\alpha + 1$ ideoque k numerus impar; at vero signum $-$ valebit, si $2k - 1$ fuerit formae $4\alpha - 1$ sive $k = 2\alpha$ ideoque numerus par. Ad valorem huius seriei inveniendum ponamus

$$\begin{aligned} S = &\left(1 - \frac{1}{2k}\right) \cos. \frac{n\pi}{2k} - \left(1 - \frac{3}{2k}\right) \cos. \frac{3n\pi}{2k} + \left(1 - \frac{5}{2k}\right) \cos. \frac{5n\pi}{2k} \\ &- \left(1 - \frac{7}{2k}\right) \cos. \frac{7n\pi}{2k} + \left(1 - \frac{9}{2k}\right) \cos. \frac{9n\pi}{2k} - \cdots \pm \left(1 - \frac{2k-1}{2k}\right) \cos. \frac{(2k-1)n\pi}{2k}, \end{aligned}$$

ita ut valor noster quaesitus sit $\frac{2\pi S}{k}$. Quo nunc valorem ipsius S investigemus, multiplicemus utrinque per $2 \cos. \frac{n\pi}{2k}$, et cum in genere sit

$$2 \cos. \frac{n\pi}{2k} \cos. \frac{(2i-1)n\pi}{2k} = \cos. \frac{in\pi}{k} + \cos. \frac{(i-1)n\pi}{k},$$

adhibita ista reductione reperietur

$$\left\{\begin{array}{l} \qquad\qquad +\left(1-\frac{1}{2k}\right)\cos.\frac{n\pi}{k}-\left(1-\frac{3}{2k}\right)\cos.\frac{2n\pi}{k} \\ +\left(1-\frac{1}{2k}\right)-\left(1-\frac{3}{2k}\right)\cos.\frac{n\pi}{k}+\left(1-\frac{5}{2k}\right)\cos.\frac{2n\pi}{k} \end{array}\right.$$
$$\left.\begin{array}{l} +\left(1-\frac{5}{2k}\right)\cos.\frac{3n\pi}{k}-\left(1-\frac{7}{2k}\right)\cos.\frac{4n\pi}{k}+\text{etc.} \\ -\left(1-\frac{7}{2k}\right)\cos.\frac{3n\pi}{k}+\left(1-\frac{9}{2k}\right)\cos.\frac{4n\pi}{k}-\text{etc.} \end{array}\right\},$$

ubi patet quemlibet terminum superiorem cum sequente inferiori in unicum coalescere, ita ut tantum primus inferior, qui est $\left(1-\frac{1}{2k}\right)$, et ultimus superior, qui est $\pm\frac{1}{2k}\cos.n\pi$, solitarii relinquantur; facta ergo hac contractione reperietur

$$2S\cos.\frac{n\pi}{2k}=1-\frac{1}{2k}\pm\frac{1}{2k}\cos.n\pi$$
$$+\frac{1}{k}\cos.\frac{n\pi}{k}-\frac{1}{k}\cos.\frac{2n\pi}{k}+\frac{1}{k}\cos.\frac{3n\pi}{k}-\cdots\mp\frac{1}{k}\cos.\frac{(k-1)n\pi}{k},$$

ubi signum superius valet, si k fuerit numerus impar, inferius autem, si k fuerit numerus par. Ponamus porro ad hanc seriem summandam

$$T=\cos.\frac{n\pi}{k}-\cos.\frac{2n\pi}{k}+\cos.\frac{3n\pi}{k}-\cos.\frac{4n\pi}{k}+\cdots\mp\cos.\frac{(k-1)n\pi}{k},$$

ita ut hoc valore T invento futurum sit

$$2S\cos.\frac{n\pi}{2k}=1-\frac{1}{2k}\pm\frac{1}{2k}\cos.n\pi+\frac{T}{k}.$$

Multiplicemus simili modo utrinque per $2\cos.\frac{n\pi}{2k}$ et in subsidium vocata eadem reductione reperietur

$$2T\cos.\frac{n\pi}{2k}$$
$$=\left\{\begin{array}{l} \qquad\qquad +\cos.\frac{3n\pi}{2k}-\cos.\frac{5n\pi}{2k}+\cos.\frac{7n\pi}{2k}-\cdots\mp\cos.\frac{(2k-1)n\pi}{2k} \\ +\cos.\frac{n\pi}{2k}-\cos.\frac{3n\pi}{2k}+\cos.\frac{5n\pi}{2k}-\cos.\frac{7n\pi}{2k}+\cdots \end{array}\right\},$$

ubi omnes termini se mutuo destruunt praeter primum inferiorem et ultimum superiorem, ita ut obtineamus

$$2T\cos.\frac{n\pi}{2k}=\cos.\frac{n\pi}{2k}\mp\cos.\frac{(2k-1)n\pi}{2k}.$$

Quia autem $\frac{2k-1}{2k}n\pi=n\pi-\frac{n\pi}{2k}$, erit

$$\cos.\frac{(2k-1)n\pi}{2k}=\cos.n\pi\cos.\frac{n\pi}{2k}+\sin.n\pi\sin.\frac{n\pi}{2k};$$

quoniam vero n supponitur numerus integer, erit $\sin.n\pi=0$ ideoque

$$2T\cos.\frac{n\pi}{2k}=\cos.\frac{n\pi}{2k}\mp\cos.n\pi\cos.\frac{n\pi}{2k},$$

unde fit

$$T=\frac{1}{2}\mp\frac{1}{2}\cos.n\pi,$$

quo valore substituto fiet

$$2S\cos.\frac{n\pi}{2k}=1,$$

consequenter

$$S=\frac{1}{2\cos.\frac{n\pi}{2k}}$$

ideoque valor noster quaesitus erit

$$\frac{\pi}{k\cos.\frac{n\pi}{2k}},$$

unde nascitur sequens

THEOREMA 1

24. *Ista formula integralis*

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extensa producit hunc valorem*

$$\frac{\pi}{k\cos.\frac{n\pi}{2k}}.$$

Cuius demonstratio ex praecedente paragrapho liquet. Huic adiungi potest sequens theorema, quod prorsus singulari demonstratione ex isto derivare licet.

THEOREMA 2

25. *Si tam ista formula integralis*

$$\int \frac{x^{k-n}}{1+x^{2k}} \cdot \frac{dx}{x}$$

quam haec

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x}$$

a termino $x=0$ *usque ad* $x=\infty$ *extendatur, utraque producet eandem summam, quae est*

$$\frac{\pi}{2k\cos.\frac{n\pi}{2k}}.$$

DEMONSTRATIO

Ponatur $S=\int \frac{x^{k-n}}{1+x^{2k}} \cdot \frac{dx}{x}$, siquidem integratio a termino $x=0$ usque ad terminum $x=\infty$ extendatur, ac ponatur $x=\frac{1}{z}$, ita ut iam integratio absolvi debeat a termino ∞ usque ad 0, et ob $\frac{dx}{x}=-\frac{dz}{z}$ habebitur nunc

$$-\int \frac{z^{-k+n}}{1+z^{-2k}} \cdot \frac{dz}{z},$$

quae, si numerator ac denominator multiplicetur per z^{2k}, abit in hanc

$$S=-\int \frac{z^{k+n}}{1+z^{2k}} \cdot \frac{dz}{z}$$

integratione a termino $z=\infty$ usque ad $z=0$ extensa. Hinc permutatis terminis integrationis erit

$$S=\int \frac{z^{k+n}}{1+z^{2k}} \cdot \frac{dz}{z}$$

a termino $z=0$ usque ad $z=\infty$; unde si loco z scribatur x, manifestum est utramque formulam a termino $x=0$ usque ad $x=\infty$ extensam eandem habere summam S. Cum igitur ambae hae formulae iunctae praebeant summam $2S=\frac{\pi}{k}:\cos.\frac{n\pi}{2k}$, erit utique utriusque formulae valor seorsim

$$S=\frac{\pi}{2k\cos.\frac{n\pi}{2k}}.$$

QUAESTIO ALTERA

26. *Proposita formula differentiali*

$$\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponatur $x=1$.

SOLUTIO

Cum in forma integralis generali quilibet terminus inventus sit

$$\frac{2}{k}\cos. n\omega \operatorname{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

fiat hic $x=1$ ac prodibit $\frac{2}{k}\cos. n\omega \operatorname{A tang.}\frac{\sin.\omega}{1-\cos.\omega}$, quae forma ob

$$\sin.\omega = 2\sin.\frac{1}{2}\omega\cos.\frac{1}{2}\omega \quad \text{et} \quad 1-\cos.\omega = 2\sin.\frac{1}{2}\omega^2$$

abit in hanc

$$\frac{2}{k}\cos. n\omega \operatorname{A tang.}\frac{\cos.\frac{1}{2}\omega}{\sin.\frac{1}{2}\omega},$$

quae, cum sit

$$\frac{\cos.\frac{1}{2}\omega}{\sin.\frac{1}{2}\omega} = \cot.\frac{1}{2}\omega = \operatorname{tang.}\left(\frac{\pi}{2}-\frac{1}{2}\omega\right),$$

porro transformatur in hanc

$$\frac{2}{k}\cos. n\omega\left(\frac{\pi}{2}-\frac{1}{2}\omega\right) = \frac{1}{k}(\pi-\omega)\cos. n\omega.$$

Quodsi igitur hic loco ω successive scribamus eius valores, qui sunt $\frac{\pi}{2k}$, $\frac{3\pi}{2k}$, $\frac{5\pi}{2k}$ usque ad $\frac{(2k-1)\pi}{2k}$, valor integralis quaesitus exprimetur per hanc progressionem

$$\frac{1}{k}\left(\pi-\frac{\pi}{2k}\right)\cos.\frac{n\pi}{2k} - \frac{1}{k}\left(\pi-\frac{3\pi}{2k}\right)\cos.\frac{3n\pi}{2k} + \frac{1}{k}\left(\pi-\frac{5\pi}{2k}\right)\cos.\frac{5n\pi}{2k}$$

$$-\frac{1}{k}\left(\pi-\frac{7\pi}{2k}\right)\cos.\frac{7n\pi}{2k} + \cdots \pm \frac{1}{k}\left(\pi-\frac{(2k-1)\pi}{2k}\right)\cos.\frac{(2k-1)n\pi}{2k},$$

ubi signorum ambiguorum superius valet, si k fuerit numerus impar, inferius vero, si par. Comparemus hanc expressionem cum ea, ad quam in quaestione praecedente est perventum, ac reperiemus hanc illius praecise esse semissem, unde eius valor erit

$$\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

sicque habebitur sequens

THEOREMA

27. *Ista formula integralis*

$$\int \frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}$$

a termino $x=0$ *usque ad terminum* $x=1$ *extensa producet hunc valorem*

$$\frac{\pi}{2k\cos.\frac{n\pi}{2k}}.$$

COROLLARIUM

28. Cum igitur huius formulae integrale a termino $x=0$ usque ad $x=1$ extensum sit dimidium eius, quod a termino $x=0$ usque ad $x=\infty$ extenditur, sequitur, si eadem formula integralis a termino $x=1$ usque ad $x=\infty$ extendatur, eius valorem quoque fore $\frac{\pi}{2k\cos.\frac{n\pi}{2k}}$; praeterea vero utriusque valor aequabitur huic formulae integrali $\int\frac{x^{k\pm n}}{1+x^{2k}}\cdot\frac{dx}{x}$, siquidem ab $x=0$ usque ad $x=\infty$ extendatur.

PROBLEMA PARTICULARE 2

29. *Si sumatur*

$$Q=1-x^{2k},$$

investigare integrale huius formulae differentialis

$$\frac{Pdx}{x(1-x^{2k})},$$

ubi quidem $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$, *ne scilicet ista fractio evadat spuria.*

SOLUTIO

Cum sit $Q = 1 - x^{2k}$, statim duo eius habentur factores simplices reales, qui sunt $1 - x$ et $1 + x$, quare partes integrales ex iis oriundas primum investigemus. Ponamus igitur pro factore $1 - x$ fractionem

$$\frac{P}{x(1 - x^{2k})} = \frac{\alpha}{1 - x} + R,$$

ubi R complectitur omnes reliquas partes, unde per $1 - x$ multiplicando habebimus

$$\frac{P(1 - x)}{x(1 - x^{2k})} = \alpha + R(1 - x);$$

quare si faciamus $x = 1$, nanciscemur

$$\alpha = \frac{P}{x} \cdot \frac{1 - x}{1 - x^{2k}},$$

cuius posterioris fractionis posito $x = 1$ tam numerator quam denominator evanescit; hinc eorum loco scribamus eorum differentialia fietque

$$\alpha = \frac{P}{x} \cdot \frac{1}{2kx^{2k-1}}.$$

Fiat igitur nunc $x = 1$, quo casu abeat P in B, ac prodibit $\alpha = \frac{B}{2k}$; ex fractione autem partiali $\frac{\alpha}{1 - x}$ porro reperitur pars integralis inde nata

$$-\alpha l(1 - x) = -\frac{B}{2k} l(1 - x).$$

Pro altero factore $1 + x$ faciamus simili modo

$$\frac{P}{x(1 - x^{2k})} = \frac{\beta}{1 + x} + R,$$

unde per $1 + x$ multiplicando fit

$$\frac{P(1 + x)}{x(1 - x^{2k})} = \beta + R(1 + x);$$

quare si faciamus $x = -1$, erit

$$\beta = \frac{P}{x} \cdot \frac{1 + x}{1 - x^{2k}},$$

ubi in posteriore fractione differentialia tam supra quam infra scribantur, ut prodeat

$$\beta = \frac{P}{x} \cdot \frac{1}{-2kx^{2k-1}} = -\frac{P}{2kx^{2k}} = -\frac{P}{2k},$$

posito scilicet $x = -1$. Ponamus ergo facto $x = -1$ functionem P abire in C fietque $\beta = -\frac{C}{2k}$ et ex fractione partiali $\frac{\beta}{1+x}$ obtinebitur pars inde nata integralis

$$\beta l(1+x) = -\frac{C}{2k} l(1+x)$$

sicque ex ambobus factoribus $1+x$ et $1-x$ nascuntur hae duae partes integrales

$$-\frac{B}{2k} l(1-x) - \frac{C}{2k} l(1+x).$$

His expeditis sit formulae $1 - x^{2k}$ factor trinominalis quicunque

$$1 - 2x \cos.\omega + xx,$$

quo facto $= 0$ ista formula $1 - x^{2k}$ induet hanc formam

$$1 - \frac{x - \cos.\omega}{\sin.\omega} \sin.2k\omega - \cos.2k\omega;$$

quae formula cum debeat evanescere, has suppeditat conditiones

$$1.\ \sin.2k\omega = 0 \quad \text{et} \quad 2.\ \cos.2k\omega = 1;$$

ex posteriore statim intelligitur angulum ω sequentes valores accipere posse

$$1.\ \omega = 0, \quad 2.\ \omega = \frac{2\pi}{2k} = \frac{\pi}{k}, \quad 3.\ \omega = \frac{4\pi}{2k} = \frac{2\pi}{k}$$

et in genere

$$\omega = \frac{i\pi}{k}.$$

Quia igitur numerus horum valorum debet esse $= k$, primus autem $\omega = 0$ tantum factori simplici respondet, numerus valorum debet sumi $k+1$, ita ut iam ultimus futurus sit $\frac{k\pi}{k} = \pi$, unde alter factor simplex $1+x$ nascitur; hoc autem modo simul primae conditioni satisfit, qua esse debet $\sin.2k\omega = 0$.

Nunc consideremus factorem generalem $xx - 2x\cos.\omega + 1$, quo posito $= 0$ fiat

$$P = \frac{F(x - \cos.\omega)}{\sin.\omega} + G,$$

eritque $\frac{x\,dQ}{dx} = -2kx^{2k}$; at vero iam vidimus tum fieri $x^{2k} = 1$ sicque $\frac{x\,dQ}{dx} = -2k$, pro qua forma in genere posuimus $\frac{f(x - \cos.\omega)}{\sin.\omega} + g$, quocirca pro nostro casu erit $f = 0$ et $g = -2k$. Cum igitur pro hoc factore in genere inventa sit ista pars integralis

$$\frac{2(Ff + Gg)}{ff + gg}\, l\sqrt{(xx - 2x\cos.\omega + 1)} + \frac{2(fG - Fg)}{ff + gg}\,\text{A tang.}\,\frac{x\sin.\omega}{1 - x\cos.\omega},$$

erit ista pars nostro casu

$$-\frac{G}{k}\, l\sqrt{(xx - 2x\cos.\omega + 1)} + \frac{F}{k}\,\text{A tang.}\,\frac{x\sin.\omega}{1 - x\cos.\omega};$$

consequenter si loco ω successive scribantur valores indicati, scilicet $\omega = 0$, $\omega = \frac{\pi}{k}$, $\omega = \frac{2\pi}{k}$ usque ad $\omega = \frac{k\pi}{k}$, et omnes istae partes in unam summam colligantur, obtinebitur totum integrale formulae propositae. Hic autem probe observandum est ex parte prima et ultima eas ipsas partes oriri, quas iam pro valoribus $1 - x$ et $1 + x$ assignavimus, quare eas penitus omitti conveniet.

COROLLARIUM 1

30. Quodsi numerator P fuerit potestas simplex ipsius x, puta x^m existente $m > 1$ et $m < 2k$, ut formula integranda sit

$$\int \frac{x^{m-1}dx}{1 - x^{2k}},$$

pro factoribus simplicibus $1 - x$ et $1 + x$ erit $B = +1$ et $C = (-1)^m$, unde partes integralis hinc natae erunt

$$\frac{-1}{2k}\, l(1 - x) - \frac{(-1)^m}{2k}\, l(1 + x).$$

Pro reliquis vero partibus erit $F = \sin.m\omega$ et $G = \cos.m\omega$, unde quaelibet

portio integralis induet hanc formam

$$-\frac{\cos. m\omega}{k} l\sqrt{(xx-2x\cos.\omega+1)}+\frac{\sin. m\omega}{k}\,\text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

ubi valores pro ω substituendi sunt $\frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \ldots \frac{(k-1)\pi}{k}$.

COROLLARIUM 2

31. Si numerator P pluribus huiusmodi terminis constet, ut sit

$$P=ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.},$$

integratio maiore difficultate non laborat; erit enim

$$F=a\sin.\alpha\omega+b\sin.\beta\omega+c\sin.\gamma\omega+\text{etc.}$$

et

$$G=a\cos.\alpha\omega+b\cos.\beta\omega+c\cos.\gamma\omega+\text{etc.}$$

hincque totum integrale facile expedietur.

SCHOLION

32. Casus prae ceteris memoratu dignus, qui hic occurrit, est, quo statuitur $P=x^{k-n}-x^{k+n}$, quippe quo omnes partes logarithmicae se mutuo tollere reperiuntur, unde eum in sequente problemate evolvamus.

PROBLEMA SPECIALE

33. *Proposita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

eius totum integrale investigare.

SOLUTIO

Quia hic est $P=x^{k-n}-x^{k+n}=x^{k-n}(1-x^{2n})$, ob n numerum integrum posito tam $x=1$ quam $x=-1$ iste valor evanescat, unde fiet tam $B=0$

quam $C = 0$, sicque partes integrales ex factoribus simplicibus natae sponte evanescunt. Pro factore autem duplici quocunque

$$1 - 2x\cos.\omega + xx$$

eo facto $= 0$ reperietur

$$P = \frac{x - \cos.\omega}{\sin.\omega}(\sin.(k-n)\omega - \sin.(k+n)\omega) + \cos.(k-n)\omega - \cos.(k+n)\omega$$

hincque colligitur

$$F = \sin.(k-n)\omega - \sin.(k+n)\omega$$

et

$$G = \cos.(k-n)\omega - \cos.(k+n)\omega.$$

Cum autem in genere sit

$$\sin.p - \sin.q = 2\sin.\frac{p-q}{2}\cos.\frac{p+q}{2}$$

et

$$\cos.p - \cos.q = 2\sin.\frac{q-p}{2}\sin.\frac{p+q}{2},$$

ob

$$p = (k-n)\omega \quad \text{et} \quad q = (k+n)\omega$$

erit

$$F = -2\sin.n\omega\cos.k\omega \quad \text{et} \quad G = 2\sin.n\omega\sin.k\omega.$$

Est vero, uti vidimus, in genere $\omega = \frac{i\pi}{k}$, unde fit

$$\sin.k\omega = \sin.i\pi = 0 \quad \text{et} \quad \cos.k\omega = \pm 1,$$

scilicet valebit $+1$, si i est numerus par, et -1, si i impar. Ad hanc autem ambiguitatem evitandam retineamus $\cos.k\omega$ atque habebimus

$$F = -2\sin.n\omega\cos.k\omega \quad \text{et} \quad G = 0.$$

Ex his igitur pars integralis quaecunque in genere erit

$$-\frac{2\sin.n\omega\cos.k\omega}{k}\,\text{A tang.}\frac{x\sin.\omega}{1 - x\cos.\omega},$$

ubi tantum opus est loco ω valores indicatos successive substitui; et quia pro primo $\omega = 0$ et ultimo $\omega = \pi$ partes integrales sponte evanescunt,

perinde est, sive valores primus et ultimus reiiciantur sive retineantur, quamobrem totum integrale quaesitum sequenti modo exprimetur:

$$\frac{2}{k}\sin.\frac{n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}-\frac{2}{k}\sin.\frac{2n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{2\pi}{k}}{1-x\cos.\frac{2\pi}{k}}$$

$$+\frac{2}{k}\sin.\frac{3n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}-\frac{2}{k}\sin.\frac{4n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{4\pi}{k}}{1-x\cos.\frac{4\pi}{k}}$$

$$\cdots\cdots\cdots$$

$$\pm\frac{2}{k}\sin.\frac{(k-1)n\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{(k-1)\pi}{k}}{1-x\cos.\frac{(k-1)\pi}{k}},$$

ubi signum superius valet, si n fuerit numerus par, inferius vero, si impar.

COROLLARIUM

34. Si hic sumere velimus $n=0$, formula integranda sponte evanescit, ita ut hic nihil memoratu dignum resultet, unde in valores huius integralis pro casibus $x=\infty$ et $x=1$ inquiramus.

QUAESTIO PRIOR

35. *Proposita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

eius valorem integralem investigare, qui oritur, si post integrationem ponitur $x=\infty$.

SOLUTIO

Cum arcuum, qui hic occurrunt, forma generalis sit A tang. $\frac{x\sin.\omega}{1-x\cos.\omega}$, ea posito $x=\infty$, uti ante vidimus, abit in $\pi-\omega$, unde litterae ω valores suos ordine tribuendo valor quaesitus nostrae formulae integralis sequenti progressione exprimetur:

$$\frac{2}{k}\left(\pi-\frac{\pi}{k}\right)\sin.\frac{n\pi}{k}-\frac{2}{k}\left(\pi-\frac{2\pi}{k}\right)\sin.\frac{2n\pi}{k}+\frac{2}{k}\left(\pi-\frac{3\pi}{k}\right)\sin.\frac{3n\pi}{k}$$
$$-\frac{2}{k}\left(\pi-\frac{4\pi}{k}\right)\sin.\frac{4n\pi}{k}+\cdots\pm\frac{2}{k}\left(\pi-\frac{(k-1)\pi}{k}\right)\sin.\frac{(k-1)n\pi}{k}.$$

Ad huius valorem investigandum ponamus

$$S=\left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{k}-\left(1-\frac{2}{k}\right)\sin.\frac{2n\pi}{k}+\left(1-\frac{3}{k}\right)\sin.\frac{3n\pi}{k}$$
$$-\left(1-\frac{4}{k}\right)\sin.\frac{4n\pi}{k}+\cdots\pm\left(1-\frac{k-1}{k}\right)\sin.\frac{(k-1)n\pi}{k},$$

ita ut valor, quem quaerimus, sit $\frac{2\pi S}{k}$. Multiplicemus igitur ut hactenus utrinque per $2\cos.\frac{n\pi}{2k}$, et cum in genere sit

$$2\sin.\frac{in\pi}{k}\cos.\frac{n\pi}{2k}=\sin.\frac{(2i+1)n\pi}{2k}+\sin.\frac{(2i-1)n\pi}{2k},$$

facta hac reductione perveniemus ad sequentem expressionem

$$2S\cos.\frac{n\pi}{2k}=\left\{\begin{array}{l}\qquad\qquad+\left(1-\frac{1}{k}\right)\sin.\frac{3n\pi}{2k}-\left(1-\frac{2}{k}\right)\sin.\frac{5n\pi}{2k}\\ \left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{2k}-\left(1-\frac{2}{k}\right)\sin.\frac{3n\pi}{2k}+\left(1-\frac{3}{k}\right)\sin.\frac{5n\pi}{2k}\end{array}\right.$$
$$\left.\begin{array}{l}+\left(1-\frac{3}{k}\right)\sin.\frac{7n\pi}{2k}-\cdots\mp\left(1-\frac{k-2}{k}\right)\sin.\frac{(2k-3)n\pi}{2k}\pm\left(1-\frac{k-1}{k}\right)\sin.\frac{(2k-1)n\pi}{2k}\\ -\left(1-\frac{4}{k}\right)\sin.\frac{7n\pi}{2k}+\cdots\pm\left(1-\frac{k-1}{k}\right)\sin.\frac{(2k-3)n\pi}{2k}\end{array}\right\},$$

ubi quilibet terminus superior cum sequente inferiori in unum contrahi potest, unde primum inferiorem cum ultimo superiori seorsim exhibeamus hoc modo

$$2S\cos.\frac{n\pi}{2k}=\left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{2k}\pm\frac{1}{k}\sin.\frac{(2k-1)n\pi}{2k}$$
$$+\frac{1}{k}\sin.\frac{3n\pi}{2k}-\frac{1}{k}\sin.\frac{5n\pi}{2k}+\frac{1}{k}\sin.\frac{7n\pi}{2k}-\cdots\mp\frac{1}{k}\sin.\frac{(2k-3)n\pi}{2k}.$$

Hoc igitur modo ultimus superior cum reliquis eandem legem sequitur, ita ut ponere liceat

$$2S\cos.\frac{n\pi}{2k}=\left(1-\frac{1}{k}\right)\sin.\frac{n\pi}{2k}$$
$$+\frac{1}{k}\sin.\frac{3n\pi}{2k}-\frac{1}{k}\sin.\frac{5n\pi}{2k}+\frac{1}{k}\sin.\frac{7n\pi}{2k}-\cdots\pm\frac{1}{k}\sin.\frac{(2k-1)n\pi}{2k}.$$

Statuamus porro

$$T = \sin.\frac{3n\pi}{2k} - \sin.\frac{5n\pi}{2k} + \sin.\frac{7n\pi}{2k} - \cdots \pm \sin.\frac{(2k-1)n\pi}{2k},$$

ut sit

$$2S\cos.\frac{n\pi}{2k} = \left(1 - \frac{1}{k}\right)\sin.\frac{n\pi}{2k} + \frac{T}{k}.$$

Iam iterum multiplicemus per $2\cos.\frac{n\pi}{2k}$ et adhibita eadem reductione reperiemus

$$2T\cos.\frac{n\pi}{2k} = \sin.\frac{2n\pi}{k} - \sin.\frac{3n\pi}{k} + \sin.\frac{4n\pi}{k} - \cdots \mp \sin.\frac{(k-1)n\pi}{k} \pm \sin.n\pi$$
$$+ \sin.\frac{n\pi}{k} - \sin.\frac{2n\pi}{k} + \sin.\frac{3n\pi}{k} - \sin.\frac{4n\pi}{k} + \cdots \pm \sin.\frac{(k-1)n\pi}{k},$$

ubi destructis terminis, qui se mutuo tollunt, obtinebitur

$$2T\cos.\frac{n\pi}{2k} = \sin.\frac{n\pi}{k} \pm \sin.n\pi = \sin.\frac{n\pi}{k}$$

ob $\sin.n\pi = 0$. Quia igitur est

$$\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k},$$

erit

$$T = \sin.\frac{n\pi}{2k},$$

quo valore substituto fiet

$$2S\cos.\frac{n\pi}{2k} = \sin.\frac{n\pi}{2k}$$

ideoque

$$S = \frac{1}{2}\operatorname{tang}.\frac{n\pi}{2k};$$

consequenter nostrae formulae integralis casu $x = \infty$ valor erit

$$\frac{\pi}{k}\operatorname{tang}.\frac{n\pi}{2k},$$

unde nascitur sequens

THEOREMA

36. *Ista formula integralis*

$$\int \frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extensa producit hunc valorem*

$$\frac{\pi}{k}\,\text{tang.}\,\frac{n\pi}{2k}.$$

SCHOLION

37. Cum haec formula duabus constet partibus, si simili modo, ut supra factum est, in valorem utriusque seorsim inquirere velimus, utriusque valor adeo imaginarius esse deprehendetur, id quod facile inde percipitur, quod posito $x=1$ ipsa fractio iam in infinitum excrescat. Tractemus autem ut supra partem priorem ponendo $S=\int\frac{x^{k-n}}{1-x^{2k}}\cdot\frac{dx}{x}$ a termino $x=0$ usque ad $x=\infty$ ac faciendo $x=\frac{1}{z}$ fiet

$$S=-\int\frac{z^{-k+n}}{1-z^{-2k}}\cdot\frac{dz}{z}=-\int\frac{z^{k+n}}{z^{2k}-1}\cdot\frac{dz}{z}$$

a termino $z=\infty$ usque ad $z=0$, ergo mutatis terminis integrationis erit

$$S=-\int\frac{z^{k+n}}{1-z^{2k}}\cdot\frac{dz}{z}$$

a termino $z=0$ usque ad $z=\infty$; unde si loco z scribamus x et has formulas iungamus, erit

$$2S=\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}=\frac{\pi}{k}\,\text{tang.}\,\frac{n\pi}{2k},$$

unde prodiret $S=\frac{\pi}{2k}\,\text{tang.}\,\frac{n\pi}{2k}$. Haec autem conclusio admitti nequit, quoniam nostra formula integralis eatenus tantum ad arcum circularem reduci poterit, quatenus numerator $x^{k-n}-x^{k+n}$ cum denominatore $1-x^{2k}$ factorem communem habet $1-xx$, qui ergo semper per divisionem tolli posset. Verum

sumta tantum alterutra parte iste factor $1-xx$ ex denominatore non tollitur ex eoque igitur necessario nasceretur pars integralis vel huius formae $\alpha l\frac{1+x}{1-x}$ vel huius $\alpha l(1-xx)$, quae utraque forma sumto $x=\infty$ fit imaginaria.

QUAESTIO ALTERA

38. *Proposita formula differentiali*

$$\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

eius integrale investigare, quod oritur, si post integrationem ponitur $x=1$.

SOLUTIO

Si in forma generali arcuum, quibus integrale exprimitur, quae est $\text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$, ponatur $x=1$, prodit, ut ante vidimus, $\frac{\pi}{2}-\frac{\omega}{2}$; qui valor cum sit dimidius eius, quem casu praecedente habuimus, statim patet valorem nostrum fore $\frac{\pi}{2k}\text{ tang.}\frac{n\pi}{2k}$, unde nascitur istud

THEOREMA

39. *Ista formula integralis*

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}$$

a termino $x=0$ *usque ad terminum* $x=1$ *extensa producet hunc valorem*

$$\frac{\pi}{2k}\text{ tang.}\frac{n\pi}{2k}.$$

COROLLARIUM

40. Hinc si eiusdem formulae integrale a termino $x=1$ usque ad $x=\infty$ extendatur, eius valor quoque erit $\frac{\pi}{2k}\text{ tang.}\frac{n\pi}{2k}$, quandoquidem hi duo valores iunctim sumti valorem casus praecedentis producere debent.

PROBLEMA PARTICULARE 3

41. *Si sumatur* $Q = 1 + 2x^k \cos.\eta + x^{2k}$, *investigare integrale huius formulae differentialis*

$$\frac{Pdx}{x(1 + 2x^k \cos.\eta + x^{2k})},$$

ubi quidem $\frac{P}{x}$ *sit functio integra, in qua nullae potestates altiores occurrant quam exponentis* $2k$.

SOLUTIO

Quia denominator Q alios factores simplices praeter imaginarios non admittit nisi casu, quo $\eta = 180^0$, sit eius factor trinomialis in genere $1 - 2x\cos.\omega + xx$, quo posito $= 0$ fiet

$$Q = \frac{x - \cos.\omega}{\sin.\omega}(\sin.2k\omega + 2\cos.\eta\sin.k\omega) + \cos.2k\omega + 2\cos.\eta\cos.k\omega + 1;$$

quae forma quia debet esse nihilo aequalis, postulat has duas conditiones

$$\text{I.}\quad \sin.2k\omega + 2\cos.\eta\sin.k\omega = 0$$

et

$$\text{II.}\quad \cos.2k\omega + 2\cos.\eta\cos.k\omega + 1 = 0.$$

Cum igitur sit

$$\sin.2k\omega = 2\sin.k\omega\cos.k\omega \quad \text{et} \quad \cos.2k\omega + 1 = 2\cos.k\omega^2,$$

prior conditio dat

$$2\sin.k\omega(\cos.k\omega + \cos.\eta) = 0$$

et secunda conditio

$$2\cos.k\omega(\cos.k\omega + \cos.\eta) = 0;$$

utrique igitur conditioni satisfit simul, si fuerit

$$\cos.k\omega + \cos.\eta = 0;$$

quod quo facilius fieri possit, sumamus $\eta = \pi - \theta$, ut habeatur $\cos.k\omega = \cos.\theta$. Omnes autem anguli cum θ communem cosinum habentes sunt θ, $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$ et in genere $2i\pi \pm \theta$, quamobrem statuamus pro ω sequentes valores

$$\omega = \frac{\theta}{k}, \quad \omega = \frac{2\pi + \theta}{k}, \quad \omega = \frac{4\pi + \theta}{k} \quad \text{etc.}$$

et in genere $\omega = \frac{2i\pi + \theta}{k}$; quorum valorum numerus cum debeat esse $= k$, ultimus erit

$$\omega = \frac{2(k-1)\pi + \theta}{k} \quad \text{sive} \quad \omega = \frac{-2\pi + \theta}{k}.$$

His constitutis consideremus formulam $\frac{x\,dQ}{dx}$, quae erit

$$= 2k(x^k \cos.\eta + x^{2k}),$$

quae per conditionem $xx - 2x\cos.\omega + 1 = 0$ ob $\cos.\eta = -\cos.\theta$ reducitur ad hanc formam

$$2k\frac{x - \cos.\omega}{\sin.\omega}(\sin.2k\omega - \cos.\theta\sin.k\omega) + 2k(\cos.2k\omega - \cos.\theta\cos.k\omega),$$

pro qua in genere sumsimus

$$= \frac{f(x - \cos.\omega)}{\sin.\omega} + g,$$

sicque erit

$$f = 2k(\sin.2k\omega - \cos.\theta\sin.k\omega)$$

et

$$g = 2k(\cos.2k\omega - \cos.\theta\cos.k\omega).$$

Loco $\sin.2k\omega$ et $\cos.2k\omega$ scribamus valores ante indicatos prodibitque

$$f = 2k\sin.k\omega(2\cos.k\omega - \cos.\theta)$$

et

$$g = 2k(2\cos.k\omega^2 - 1 - \cos.\theta\cos.k\omega);$$

cum autem esse debeat $\cos.k\omega = \cos.\theta$, fiet

$$f = 2k\sin.k\omega\cos.\theta = k\sin.2k\omega$$

et

$$g = 2k(\cos.\theta^2 - 1) = -2k\sin.\theta^2.$$

Quia igitur in genere est $\omega = \frac{2i\pi + \theta}{k}$, erit $2k\omega = 4i\pi + 2\theta$, ita ut iam habeamus $f = k\sin.2\theta = 2k\sin.\theta\cos.\theta$, ita ut sit $ff + gg = 4kk\sin.\theta^2$; quocirca si posito $1 - 2x\cos.\omega + xx = 0$ functio P transformetur in hanc formam $\frac{F(x - \cos.\omega)}{\sin.\omega} + G$, ex denominatoris Q factore $1 - 2x\cos.\omega + xx$ orietur ista pars integralis

$$\frac{F\cos.\theta - G\sin.\theta}{k\sin.\theta}\, l\sqrt{(xx - 2x\cos.\omega + 1)} + \frac{G\cos.\theta + F\sin.\theta}{k\sin.\theta}\,\text{A tang.}\frac{x\sin.\omega}{1 - x\cos.\omega}.$$

Tantum igitur superest, ut loco ω ordine substituantur omnes eius valores, qui sunt $\frac{\theta}{k}$, $\frac{2\pi+\theta}{k}$, $\frac{4\pi+\theta}{k}$, $\ldots \frac{2(k-1)\pi+\theta}{k}$, et summa omnium harum formularum praebebit totum integrale quaesitum.

COROLLARIUM

42. Si fuerit numerator P simplex potestas ipsius x, scilicet $P = x^m$, tum fiet $F = \sin. m\omega$ et $G = \cos. m\omega$, unde pars integralis ex denominatoris factore indefinito $1 - 2x\cos.\omega + xx$ oriunda erit

$$\frac{\cos.\theta \sin. m\omega - \sin.\theta \cos. m\omega}{k \sin.\theta} \, l\sqrt{(xx - 2x\cos.\omega + 1)}$$
$$+ \frac{\cos.\theta \cos. m\omega + \sin.\theta \sin. m\omega}{k \sin.\theta} \, \text{A tang.} \frac{x \sin.\omega}{1 - x\cos.\omega},$$

unde simul patet, si functio P ex pluribus huiusmodi potestatibus fuerit composita, quemadmodum integrationem absolvi oporteat.

PROBLEMA SPECIALE

43. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + 2x^k \cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

eius totum integrale investigare.

SOLUTIO

Cum hic sit $P = x^{k-n} + x^{k+n}$, erit

$$F = 2\sin. k\omega \cos. n\omega \quad \text{et} \quad G = 2\cos. k\omega \cos. n\omega,$$

ubi $\cos. k\omega = \cos.\theta$; quibus valoribus substitutis pro parte integralis logarithmica erit

$$\frac{F\cos.\theta - G\sin.\theta}{k\sin.\theta} = \frac{2\cos.\theta \cos. n\omega (\sin. k\omega - \sin.\theta)}{k\sin.\theta}.$$

Cum autem in genere sit $\omega = \frac{2i\pi+\theta}{k}$, erit $\sin. k\omega = \sin.\theta$, unde patet hanc formulam evanescere, ita ut omnes partes logarithmicae ex integrali excedant.

Pro partibus autem circularibus evadet coefficiens

$$\frac{G\cos.\theta+F\sin.\theta}{k\sin.\theta}=\frac{2\cos.n\omega}{k\sin.\theta}$$

sicque ex factore denominatoris indefinito $1-2x\cos.\omega+xx$ oritur ista pars integralis

$$\frac{2\cos.n\omega}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.\omega}{1-x\cos.\omega}.$$

In hac ergo formula pro ω ordine scribamus eius valores, qui sunt $\frac{\theta}{k}$, $\frac{2\pi+\theta}{k}$, $\frac{4\pi+\theta}{k}$ etc. usque ad $\frac{2(k-1)\pi+\theta}{k}$, ubi meminisse oportet esse $\theta=\pi-\eta$, et quo formulae non nimis fiant perplexae, utamur sequentibus valoribus

$$\frac{2\pi}{k}=\alpha,\quad \frac{\theta}{k}=\beta,\quad \frac{2n\pi}{k}=\gamma \quad\text{et}\quad \frac{n\theta}{k}=\delta,$$

ut valores ipsius ω fiant

$$\beta,\ \alpha+\beta,\ 2\alpha+\beta,\ 3\alpha+\beta,\ \ldots\ (k-1)\alpha+\beta.$$

At vero omnes valores anguli $n\omega$ erunt ordine

$$\delta,\ \gamma+\delta,\ 2\gamma+\delta,\ 3\gamma+\delta,\ \ldots\ (k-1)\gamma+\delta.$$

His igitur valoribus adhibitis totum integrale, quod quaerimus, erit

$$\frac{2\cos.\delta}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.\beta}{1-x\cos.\beta}+\frac{2\cos.(\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.(\alpha+\beta)}{1-x\cos.(\alpha+\beta)}$$
$$+\frac{2\cos.(2\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.(2\alpha+\beta)}{1-x\cos.(2\alpha+\beta)}+\frac{2\cos.(3\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.(3\alpha+\beta)}{1-x\cos.(3\alpha+\beta)}$$
$$+\cdots+\frac{2\cos.((k-1)\gamma+\delta)}{k\sin.\theta}\,\mathrm{A\,tang.}\frac{x\sin.((k-1)\alpha+\beta)}{1-x\cos.((k-1)\alpha+\beta)}.$$

COROLLARIUM 1

44. Evolvamus casum, quo $n=0$ ideoque etiam $\gamma=0$ et $\delta=0$, et quia hoc casu formula integralis evadet

$$2\int\frac{x^{k-1}dx}{1+2x^k\cos.\eta+x^{2k}},$$

pro ea statuamus $x^k = z$ atque ob $x^{k-1}dx = \frac{dz}{k}$ habebitur

$$\frac{2}{k}\int\frac{dz}{1+2z\cos.\eta+zz},$$

cuius integrale erit

$$=\frac{2}{k\sin.\eta}\,\mathrm{A\ tang.}\frac{z\sin.\eta}{1+z\cos.\eta}=\frac{2}{k\sin.\theta}\,\mathrm{A\ tang.}\frac{z\sin.\theta}{1-z\cos.\theta};$$

unde cum in serie inventa omnes coefficientes arcuum fiant $\frac{2}{k\sin.\theta}$, per hunc coefficientem dividendo habebimus sequentem aequationem

$$\mathrm{A\ tang.}\frac{x^k\sin.\theta}{1+x^k\cos.\theta}=\mathrm{A\ tang.}\frac{x\sin.\beta}{1-x\cos.\beta}+\mathrm{A\ tang.}\frac{x\sin.(\alpha+\beta)}{1-x\cos.(\alpha+\beta)}$$
$$+\mathrm{A\ tang.}\frac{x\sin.(2\alpha+\beta)}{1-x\cos.(2\alpha+\beta)}+\cdots+\mathrm{A\ tang.}\frac{x\sin.((k-1)\alpha+\beta)}{1-x\cos.((k-1)\alpha+\beta)},$$

ubi recordandum est esse $\alpha=\frac{2\pi}{k}$, $\beta=\frac{\theta}{k}$.

COROLLARIUM 2

45. Ponamus esse $\theta = 90^0 = \frac{\pi}{2}$ et aequatio modo inventa hanc induet formam

$$\mathrm{A\ tang.}x^k=\mathrm{A\ tang.}\frac{x\sin.\frac{\pi}{2k}}{1-x\cos.\frac{\pi}{2k}}+\mathrm{A\ tang.}\frac{x\sin.\frac{5\pi}{2k}}{1-x\cos.\frac{5\pi}{2k}}+\mathrm{A\ tang.}\frac{x\sin.\frac{9\pi}{2k}}{1-x\cos.\frac{9\pi}{2k}}$$
$$+\mathrm{A\ tang.}\frac{x\sin.\frac{13\pi}{2k}}{1-x\cos.\frac{13\pi}{2k}}+\cdots+\mathrm{A\ tang.}\frac{x\sin.\frac{(4k-3)\pi}{2k}}{1-x\cos.\frac{(4k-3)\pi}{2k}}.$$

Sit $k=1$ eritque $\mathrm{A\ tang.}x=\mathrm{A\ tang.}x$.

Sit $k=2$ eritque

$$\mathrm{A\ tang.}xx=\mathrm{A\ tang.}\frac{x}{\sqrt{2}-x}+\mathrm{A\ tang.}\frac{-x}{\sqrt{2}+x}=\mathrm{A\ tang.}\frac{x}{\sqrt{2}-x}-\mathrm{A\ tang.}\frac{x}{\sqrt{2}+x}.$$

Sit $k=3$ eritque

$$\mathrm{A\ tang.}x^3=\mathrm{A\ tang.}\frac{x}{2-x\sqrt{3}}+\mathrm{A\ tang.}\frac{x}{2+x\sqrt{3}}-\mathrm{A\ tang.}x$$

etc.

Haec igitur series ab ea, quam supra (§ 20) invenimus, prorsus discrepat, etiamsi utriusque valor sit idem, scilicet $\mathrm{A\ tang.}x^k$.

QUAESTIO PRIOR

46. *Proposita formula differentiali*

$$\frac{x^{k-n}+x^{k+n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponitur $x=1$.

SOLUTIO

Cum posito $x=\infty$ in genere sit

$$\text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega}=\pi-\omega,$$

valor integralis, quem quaerimus, hac serie exprimetur

$$\begin{aligned}&\frac{2\cos.\delta}{k\sin.\theta}(\pi-\beta)+\frac{2\cos.(\gamma+\delta)}{k\sin.\theta}(\pi-\alpha-\beta)+\frac{2\cos.(2\gamma+\delta)}{k\sin.\theta}(\pi-2\alpha-\beta)\\&+\frac{2\cos.(3\gamma+\delta)}{k\sin.\theta}(\pi-3\alpha-\beta)+\cdots+\frac{2\cos.((k-1)\gamma+\delta)}{k\sin.\theta}(\pi-(k-1)\alpha-\beta).\end{aligned}$$

Statuamus igitur

$$\begin{aligned}S=(\pi-\beta)\cos.\delta+(\pi-\alpha-\beta)\cos.(\gamma+\delta)+(\pi-2\alpha-\beta)\cos.(2\gamma+\delta)\\+\cdots+(\pi-(k-1)\alpha-\beta)\cos.((k-1)\gamma+\delta),\end{aligned}$$

ut sit valor quaesitus $\frac{2S}{k\sin.\theta}$. Multiplicemus utrinque per $2\sin.\frac{1}{2}\gamma$, et cum sit

$$2\sin.\frac{1}{2}\gamma\cos.q=\sin.\left(\frac{1}{2}\gamma+q\right)-\sin.\left(q-\frac{1}{2}\gamma\right),$$

hac reductione adhibita fiet

$$2S\sin.\frac{1}{2}\gamma=\left\{\begin{aligned}&-(\pi-\beta)\sin.\left(\delta-\frac{1}{2}\gamma\right)\\&+(\pi-\beta)\sin.\left(\frac{1}{2}\gamma+\delta\right)-(\pi-\alpha-\beta)\sin.\left(\frac{1}{2}\gamma+\delta\right)\\&+(\pi-\alpha-\beta)\sin.\left(\frac{3}{2}\gamma+\delta\right)-(\pi-2\alpha-\beta)\sin.\left(\frac{3}{2}\gamma+\delta\right)\\&+(\pi-2\alpha-\beta)\sin.\left(\frac{5}{2}\gamma+\delta\right)-(\pi-3\alpha-\beta)\sin.\left(\frac{5}{2}\gamma+\delta\right)\\&+\text{etc.},\end{aligned}\right.$$

quae series contractis terminis similibus transit in hanc

$$2S\sin.\tfrac{1}{2}\gamma = -(\pi-\beta)\sin.\left(\delta-\tfrac{1}{2}\gamma\right)+\alpha\sin.\left(\tfrac{1}{2}\gamma+\delta\right)$$
$$+\alpha\sin.\left(\tfrac{3}{2}\gamma+\delta\right)+\alpha\sin.\left(\tfrac{5}{2}\gamma+\delta\right)+\cdots+\alpha\sin.\left(\tfrac{2k-3}{2}\gamma+\delta\right)$$
$$+(\pi-(k-1)\alpha-\beta)\sin.\left(\tfrac{2k-1}{2}\gamma+\delta\right);$$

ubi cum sit $\alpha=\frac{2\pi}{k}$ et $\beta=\frac{\theta}{k}$, erit

$$\pi-(k-1)\alpha-\beta=\alpha-\pi-\beta.$$

Ponatur

$$T=\sin.\left(\tfrac{1}{2}\gamma+\delta\right)+\sin.\left(\tfrac{3}{2}\gamma+\delta\right)+\sin.\left(\tfrac{5}{2}\gamma+\delta\right)$$
$$+\sin.\left(\tfrac{7}{2}\gamma+\delta\right)+\cdots+\sin.\left(\left(k-\tfrac{1}{2}\right)\gamma+\delta\right),$$

ut nanciscamur

$$2S\sin.\tfrac{1}{2}\gamma=-(\pi-\beta)\sin.\left(\delta-\tfrac{1}{2}\gamma\right)-(\pi+\beta)\sin.\left(\left(k-\tfrac{1}{2}\right)\gamma+\delta\right)+\alpha T,$$

quae expressio ob $k\gamma=2n\pi$ reducitur ad

$$-2\pi\sin.\left(\delta-\tfrac{1}{2}\gamma\right)+\alpha T.$$

Nunc igitur ad quantitatem T inveniendam multiplicemus utrinque per $2\sin.\frac{1}{2}\gamma$, et cum in genere sit

$$2\sin.\tfrac{1}{2}\gamma\sin.q=\cos.\left(q-\tfrac{1}{2}\gamma\right)-\cos.\left(q+\tfrac{1}{2}\gamma\right),$$

obtinebimus

$$2T\sin.\tfrac{1}{2}\gamma=\cos.\delta$$
$$-\cos.(\gamma+\delta)-\cos.(2\gamma+\delta)-\cos.(3\gamma+\delta)-\cos.(4\gamma+\delta)-\cdots$$
$$+\cos.(\gamma+\delta)+\cos.(2\gamma+\delta)+\cos.(3\gamma+\delta)+\cos.(4\gamma+\delta)+\cdots$$
$$-\cos.(k\gamma+\delta),$$

quae forma contrahitur in istam

$$2T\sin.\tfrac{1}{2}\gamma=\cos.\delta-\cos.(k\gamma+\delta).$$

Cum autem sit $\gamma = \frac{2n\pi}{k}$, erit $k\gamma = 2n\pi$ ideoque $\cos.(k\gamma + \delta) = \cos.\delta$, unde fit

$$2\,T\sin.\frac{1}{2}\gamma = 0,$$

ita ut nunc sit

$$2\,S\sin.\frac{1}{2}\gamma = 2\pi\sin.\left(\frac{1}{2}\gamma - \delta\right)$$

ideoque

$$S = \frac{\pi\sin.\left(\frac{1}{2}\gamma - \delta\right)}{\sin.\frac{1}{2}\gamma}.$$

Est vero $\frac{1}{2}\gamma = \frac{n\pi}{k}$ et $\delta = \frac{n\theta}{k}$ ideoque $\frac{1}{2}\gamma - \delta = \frac{n(\pi - \theta)}{k} = \frac{n\eta}{k}$ ob $\theta = \pi - \eta$ hocque modo habebimus

$$S = \frac{\pi\sin.\frac{n\eta}{k}}{\sin.\frac{n\pi}{k}};$$

consequenter valor integralis quaesiti concluditur fore

$$\frac{2\,\pi\sin.\frac{n\eta}{k}}{k\sin.\theta\sin.\frac{n\pi}{k}} = \frac{2\,\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

unde formetur sequens theorema.

THEOREMA 1

47. *Haec formula integralis*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + 2x^k\cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

a termino $x = 0$ *usque ad* $x = \infty$ *extensa producit hunc valorem*

$$\frac{2\,\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}.$$

Cui adiungatur adhuc sequens

THEOREMA 2

48. *Ista vero formula integralis*

$$\int \frac{x^{k \pm n - 1}\, dx}{1 + 2x^k \cos.\eta + x^{2k}}$$

pariter a termino $x = 0$ *usque ad terminum* $x = \infty$ *extensa valorem habet dimidium praecedentis, qui ergo erit*

$$\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}.$$

Cuius demonstratio perinde succedet ac supra (§ 25).

QUAESTIO ALTERA

49. *Proposita formula differentiali*

$$\frac{x^{k-n} + x^{k+n}}{1 + 2x^k \cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

eius integralis valorem investigare, qui oritur, si post integrationem ponitur $x = 1$.

SOLUTIO

Posito $x = 1$ formula generalis A tang. $\frac{x \sin.\omega}{1 - x \cos.\omega}$, ut supra vidimus, reducitur ad $\frac{\pi - \omega}{2}$; unde patet singulas partes integralis duplo minores esse quam casu praecedente, unde valor quaesitus etiam erit duplo minor

$$= \frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}},$$

unde nascitur sequens

THEOREMA

50. *Ista formula integralis*

$$\int \frac{x^{k-n} + x^{k+n}}{1 + 2x^k \cos.\eta + x^{2k}} \cdot \frac{dx}{x}$$

a termino $x=0$ usque ad $x=1$ extensa producet hunc valorem

$$\frac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}.$$

SCHOLION

51. In his valoribus integralibus ii casus praecipue sunt notatu digni, quibus post integrationem statuitur $x=1$, quandoquidem tum ista integralia commode per seriem infinitam exprimere licet. Ita pro casu § 26, quoniam est

$$\frac{1}{1+x^{2k}} = 1 - x^{2k} + x^{4k} - x^{6k} + \text{etc.},$$

si hanc seriem multiplicemus per $(x^{k-n}+x^{k+n})\frac{dx}{x}$ et integremus, tum vero ponamus $x=1$, prodibit ista series infinita

$$\begin{aligned}&\frac{1}{k-n} - \frac{1}{3k-n} + \frac{1}{5k-n} - \frac{1}{7k-n} + \text{etc.}\\ &+\frac{1}{k+n} - \frac{1}{3k+n} + \frac{1}{5k+n} - \frac{1}{7k+n} + \text{etc.},\end{aligned}$$

cuius ergo seriei in infinitum continuatae summa est

$$\frac{\pi}{2k \cos. \frac{n\pi}{2k}}.$$

At pro casu § 38 ob

$$\frac{1}{1-x^{2k}} = 1 + x^{2k} + x^{4k} + x^{6k} + \text{etc.}$$

eodem modo operando pervenitur ad hanc seriem

$$\begin{aligned}&\frac{1}{k-n} + \frac{1}{3k-n} + \frac{1}{5k-n} + \frac{1}{7k-n} + \text{etc.}\\ &-\frac{1}{k+n} - \frac{1}{3k+n} - \frac{1}{5k+n} - \frac{1}{7k+n} - \text{etc.},\end{aligned}$$

cuius ergo summa erit

$$\frac{\pi}{2k} \text{tang.} \frac{n\pi}{2k}.$$

Denique pro casu, quem extremo loco tractavimus, cum sit, ut alibi[1]) ostendimus,

$$\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}=\sin.\eta-x^k\sin.2\eta+x^{2k}\sin.3\eta-\text{etc.},$$

haec series ducta in $(x^{k-n}+x^{k+n})\frac{dx}{x}$ et integrata sumendo $x=1$ producet hanc seriem

$$\begin{aligned}&\frac{\sin.\eta}{k-n}-\frac{\sin.2\eta}{2k-n}+\frac{\sin.3\eta}{3k-n}-\frac{\sin.4\eta}{4k-n}+\text{etc.}\\+&\frac{\sin.\eta}{k+n}-\frac{\sin.2\eta}{2k+n}+\frac{\sin.3\eta}{3k+n}-\frac{\sin.4\eta}{4k+n}+\text{etc.},\end{aligned}$$

cuius ergo valor aequabitur illi, quem invenimus, valori ducto in $\sin.\eta$, ita ut summa huius seriei sit

$$=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}},$$

quae series eo magis sunt memorabiles, quod alio modo earum summa vix elici potest.

1) L. EULERI Commentatio 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421, imprimis p. 441. A. G.

OBSERVATIONES IN ALIQUOT THEOREMATA ILLUSTRISSIMI DE LA GRANGE

Commentatio 587 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 16—41

Postquam aliquod theorema ex iis, quae non ita pridem demonstravi, quo ostendi[1]) formulae integralis $\int \frac{(x-1)dx}{lx}$, si post integrationem ponatur $x=1$, valorem esse $=l2$, cum illustri Domino DE LA GRANGE communicassem[2]), is novitate huius argumenti permotus non solum felicissimo successu eius demonstrationem penetravit, sed etiam plurima alia praeclara inventa inde deduxit, quorum uberior enucleatio scientiae analyticae maxima incrementa polliceri videtur, ex quo genere aliquot praeclarissima specimina mecum benevole communicavit, quae statim summo studio sum perscrutatus; et quoniam haec materia attentionem mereri videtur, meas meditationes, quae se mihi hac occasione obtulerunt, fusius sum expositurus. Cum autem hoc quasi novum Analyseos genus potissimum in eiusmodi formulis integralibus versetur, in quibus variabili post integrationem certus valor determinatus tribuitur, ad taediosas verborum ambages evitandas, quas perpetua talium conditionum commemoratio postularet, peculiarem signandi modum adhibebo, quem ante omnia accuratius explicare necesse erit.

1) Vide § 5 Commentationis 464 supra, p. 155, laudatae. A. G.

2) Vide epistolas ab EULERO d. 26. Jan. et 23. Mart. 1775 ad I. L. LAGRANGE scriptas, *LEONHARDI EULERI Opera postuma* 1, p. 585 et 586; *Oeuvres de LAGRANGE* 14, p. 240 et 241; *LEONHARDI EULERI Opera omnia*, series III. A. G.

HYPOTHESIS

1. Hac signandi ratione

$$\int P dx \left[\begin{matrix}\text{ab } x=a\\ \text{ad } x=b\end{matrix}\right]$$

declaratur integrale $\int P dx$ ita esse assumtum, ut evanescat posito $x=a$, tum vero statui $x=b$; quo pacto manifestum est eius valorem penitus fore determinatum.

SCHOLION

2. Quo indoles huius determinationis clarius perspiciatur, quoniam P denotat functionem aliquam ipsius x, eius naturam repraesentemus linea quadam curva $ixabco$ (Fig. 1) super axe IO exstructa, cuius quaecunque appli-

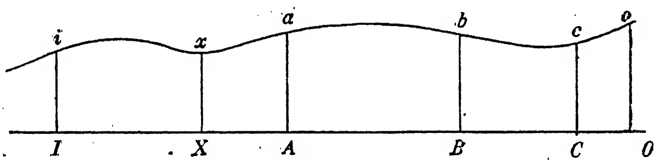

Fig. 1.

cata Xx abscissae $IX=x$ respondens exhibeat ipsam functionem P, ita ut formula integralis $\int P dx$ indefinite exprimat aream huius curvae. Quodsi iam capiantur abscissae $IA=a$, $IB=b$, quibus respondeant applicatae Aa et Bb, formula proposita exprimet aream $AaBb$ inter applicatas Aa et Bb interceptam. Eodem modo, si alia quaepiam abscissa statuatur $IC=c$, area $AaCc$ exprimetur hac formula

$$\int P dx \left[\begin{matrix}\text{ab } x=a\\ \text{ad } x=c\end{matrix}\right],$$

area autem $BbCc$ ista formula

$$\int P dx \left[\begin{matrix}\text{ab } x=b\\ \text{ad } x=c\end{matrix}\right];$$

tum vero ab initio I incipiendo area $IiAa$ indicabitur per hanc formulam

$$\int P dx \left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=a\end{matrix}\right];$$

unde sponte fluunt sequentia lemmata ita succincte expressa.

LEMMA 1

3. $$\int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} = -\int Pdx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=a \end{bmatrix}.$$

Quoniam enim, si b ut maius spectetur quam a, formula posterior

$$\int Pdx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=a \end{bmatrix}$$

eandem aream $AaBb$ refert quam prior, sed ordine retrogrado, ista expressio pro negativa erit habenda sicque erit quoque

$$\int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} + \int Pdx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=a \end{bmatrix} = 0.$$

LEMMA 2

4. $$\int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} + \int Pdx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix} = \int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=c \end{bmatrix},$$

quemadmodum inspectio figurae manifesto declarat.

LEMMA 3

5. $$\int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=c \end{bmatrix} - \int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} = \int Pdx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix},$$

ubi in binis prioribus formulis idem occurrit terminus *a quo*, scilicet $x=a$; terminorum vero *ad quem*, scilicet $x=c$ et $x=b$, posterior $x=b$ dat pro tertia formula terminum *a quo*, prior vero terminum *ad quem*.

LEMMA 4

6. $$\int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=c \end{bmatrix} - \int Pdx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix} = \int Pdx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix},$$

ubi notetur binas formulas priores eundem habere terminum *ad quem*, scilicet $x=c$, terminorum autem *a quo* priorem $x=a$ dare in tertia formula terminum *a quo*, posteriorem vero terminum *ad quem*.

LEMMA 5

7. $$\int P\,dx \begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix} + \int P\,dx \begin{bmatrix}\text{ab } x=b\\ \text{ad } x=c\end{bmatrix} + \int P\,dx \begin{bmatrix}\text{ab } x=c\\ \text{ad } x=a\end{bmatrix} = 0.$$

SCHOLION

8. His igitur, quae per se sunt maxime perspicua, praemissis argumenta praecipua, quae Celeb. DE LA GRANGE mihi perscripsit, ordine percurram. Primo autem mentionem insignis paradoxi facit, cuius indolem ipse non satis perspicere fatetur, a quo igitur meas meditationes inchoabo.

RESOLUTIO INSIGNIS PARADOXI

9. Cum Vir celeb. etiam invenisset hoc theorema generale

$$\int \frac{x^n - x^m}{lx}\cdot\frac{dx}{x} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{n}{m},$$

cuius veritatem non ita pridem pluribus demonstrationibus adstruxi[1]), posuit $x^n = z$ et $x^m = y$; quo facto pars prior $\int \frac{x^{n-1}dx}{lx}$ transformatur in hanc $\int \frac{dz}{lz}$, simili vero modo altera $\int \frac{x^{m-1}dx}{lx}$ in hanc $\int \frac{dy}{ly}$; unde his partibus seorsim positis sequitur fore

$$\int \frac{dz}{lz} \begin{bmatrix}\text{a } z=0\\ \text{ad } z=1\end{bmatrix} - \int \frac{dy}{ly} \begin{bmatrix}\text{ab } y=0\\ \text{ad } y=1\end{bmatrix} = l\frac{n}{m}.$$

Quare cum hae duae formulae omnino sint similes atque iisdem terminis integrationis contentae, quis non crederet eos etiam inter se perfecte fore aequales sive esse

$$\int \frac{dz}{lz} \begin{bmatrix}\text{a } z=0\\ \text{ad } z=1\end{bmatrix} = \int \frac{dy}{ly} \begin{bmatrix}\text{ab } y=0\\ \text{ad } y=1\end{bmatrix}?$$

Interim tamen vidimus differentiam inter has formulas esse $l\frac{n}{m}$. Hic igitur se offert quaestio maximi momenti, quemadmodum istam manifestam contradictionem dirimere oporteat.

1) Vide § 6 Commentationis 464 supra, p. 155, laudatae. A. G.

10. Primo autem hic observari convenit ambas quantitates y et z certo quodam modo a se invicem pendere. Cum enim sit $y = x^m$ et $z = x^n$, erit $y^n = z^m$, quo tamen nexu non impeditur, quominus posito sive $y = 0$ sive $y = 1$ etiam fiat $z = 0$ sive $z = 1$. Interim tamen hinc neutiquam patet, cur ob hanc rationem istae binae formulae

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab } & y = 0 \\ \text{ad } & y = 1 \end{bmatrix} \quad \text{et} \quad \int \frac{dz}{lz} \begin{bmatrix} \text{a } & z = 0 \\ \text{ad } & z = 1 \end{bmatrix}$$

dispares prodire queant; unde haec observatio ad dubium solvendum nihil plane conferre videtur.

11. Quin etiam nullo prorsus dubio obnoxia videtur haec aequatio multo generalior

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab } & y = a \\ \text{ad } & y = b \end{bmatrix} = \int \frac{dz}{lz} \begin{bmatrix} \text{a } & z = a \\ \text{ad } & z = b \end{bmatrix},$$

quandoquidem nihil plane impedit, quominus loco z scribamus y vel vicissim; verum plurima phaenomena in analysi observata satis luculenter docent huiusmodi aequalitates interdum exceptionem pati, quando valores evadunt infiniti. Haec autem circumstantia nostro casu utique locum habet, cum formula integralis $\int \frac{dy}{ly}$, si ab $y = 0$ ad $y = 1$ extendatur, utique in infinitum excrescat, quod etiam de altera $\int \frac{dz}{lz}$ est tenendum. Si enim [abscissa] fiat $= 1$, applicata nostrae curvae, quae est $\frac{1}{lz}$, manifesto fit infinite magna, unde superior aequalitas generalis

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab } & y = a \\ \text{ad } & y = b \end{bmatrix} - \int \frac{dz}{lz} \begin{bmatrix} \text{a } & z = a \\ \text{ad } & z = b \end{bmatrix} = 0$$

hanc restrictionem postulat, nisi vel a sit $= 1$ vel $b = 1$, quippe quibus casibus utraque formula fit infinita.

12. His perpensis nullum plane dubium mihi quidem superesse videtur, quin in hac circumstantia vera solutio propositi paradoxi sit quaerenda, quae scilicet in eo versatur, quod sit

$$\text{tam} \quad \int \frac{dy}{ly} \begin{bmatrix} \text{ab } & y = 0 \\ \text{ad } & y = 1 \end{bmatrix} = \infty \quad \text{quam} \quad \int \frac{dz}{lz} \begin{bmatrix} \text{a } & z = 0 \\ \text{ad } & z = 1 \end{bmatrix} = \infty,$$

ita ut horum infinitorum differentia possit aequari quantitati finitae cuicunque ideoque in se spectata prorsus non determinetur; quod autem ista differentia nostro casu sit $l\frac{n}{m}$ ideoque determinata, inde venit, quod sit $y^n = z^m$.

13. Simile aliquid evenire potest in formulis simplicioribus, quales sunt $\int\frac{dy}{y}$ et $\int\frac{dz}{z}$, quippe quarum valores a termino $y=0$ et $z=0$ sumti sunt infiniti, unde, etiamsi post integrationem idem terminus *ad quem* statuatur, scilicet $y=1$ et $z=1$, tamen hinc nullo modo sequitur differentiam absolute nihilo aequari, quin potius tanquam indeterminata spectari debebit, cum quidem pro aliis terminis integrationis certo sit

$$\int\frac{dy}{y}\begin{bmatrix}\text{ab} & y=a\\ \text{ad} & y=b\end{bmatrix}=\int\frac{dz}{z}\begin{bmatrix}\text{a} & z=a\\ \text{ad} & z=b\end{bmatrix},$$

dummodo neque a neque b fuerit $=0$ vel $=\infty$.

14. Atque hinc etiam paradoxon proposito penitus simile proferri potest, quod ita se habet

$$\int\frac{dz}{z}\begin{bmatrix}\text{a} & z=0\\ \text{ad} & z=\infty\end{bmatrix}-\int\frac{dy}{y}\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y=\infty\end{bmatrix}=la;$$

cuius veritas cum in aprico sit posita, siquidem accipiatur $z=ay$, etiam paradoxon propositum rite dilutum erit censendum.

OBSERVATIONES IN HOC THEOREMA D. DE LA GRANGE

$$\int\frac{x^n-x^m}{lx}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab} & x=a\\ \text{ad} & x=b\end{bmatrix}=\int(b^y-a^y)\frac{dy}{y}\begin{bmatrix}\text{ab} & y=m\\ \text{ad} & y=n\end{bmatrix}$$

15. Cum equidem ante aliquod tempus reductiones huiusmodi formularum tractassem, alios terminos integrationis praeterquam ab $x=0$ ad $x=1$ non sum contemplatus, unde hoc theorema mihi statim altioris indaginis est visum atque omnino dignum, quod summa cura expendatur. Primum igitur in eius veritatem per series inquirere constitui, quod negotium sequenti modo peregi.

16. Cum sit

$$x^\alpha=e^{\alpha lx}=1+\alpha lx+\frac{(\alpha lx)^2}{1\cdot 2}+\frac{(\alpha lx)^3}{1\cdot 2\cdot 3}+\text{etc.},$$

erit

$$x^n-x^m=(n-m)\frac{lx}{1}+(n^2-m^2)\frac{(lx)^2}{1\cdot 2}+\frac{(n^3-m^3)(lx)^3}{1\cdot 2\cdot 3}+\text{etc.}$$

Hanc ergo seriem ducamus in $\frac{dx}{xlx}$, et quia in genere

$$\int (lx)^{\lambda} \frac{dx}{xlx} \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} = \frac{(lb)^{\lambda}-(la)^{\lambda}}{\lambda},$$

formulae ad sinistram partem scriptae valor per hanc seriem infinitam exprimetur

$$\frac{n-m}{1}\cdot\frac{lb-la}{1}+\frac{n^2-m^2}{1\cdot 2}\cdot\frac{(lb)^2-(la)^2}{2}+\frac{n^3-m^3}{1\cdot 2\cdot 3}\cdot\frac{(lb)^3-(la)^3}{3}+\text{etc.}$$

17. Simili modo pro formula ad dextram posita per seriem infinitam erit

$$b^y-a^y=y\,\frac{lb-la}{1}+y^2\,\frac{(lb)^2-(la)^2}{1\cdot 2}+y^3\,\frac{(lb)^3-(la)^3}{1\cdot 2\cdot 3}+\text{etc.},$$

quae ergo ducatur in $\frac{dy}{y}$, et quia in genere est

$$\int y^{\lambda}\frac{dy}{y}\begin{bmatrix} \text{ab } y=m \\ \text{ad } y=n \end{bmatrix} = \frac{n^{\lambda}-m^{\lambda}}{\lambda},$$

valor istius formulae per seriem hanc infinitam exprimetur

$$\frac{n-m}{1}\cdot\frac{lb-la}{1}+\frac{n^2-m^2}{2}\cdot\frac{(lb)^2-(la)^2}{1\cdot 2}+\frac{n^3-m^3}{3}\cdot\frac{(lb)^3-(la)^3}{1\cdot 2\cdot 3}+\text{etc.}$$

Quia igitur haec series cum praecedente perfecte congruit, veritas theorematis firmiter est evicta.

18. Verum hinc neutiquam perspicitur, quomodo sagacissimus Auctor ad hoc theorema sit perductus, quamobrem rebus probe perpensis viam inveni ex iisdem principiis, quibus antehac[1]) sum usus, ad easdem formulas perveniendi. Inchoandum autem est ab hac forma simplicissima

$$\int x^{\lambda}\frac{dx}{x}\begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} = \frac{b^{\lambda}-a^{\lambda}}{\lambda},$$

ubi utrinque per $d\lambda$ multiplicans denuo integrationem instituo, et cum, uti iam passim[2]) demonstratum reperitur, sit

$$\int d\lambda \int x^{\lambda}\frac{dx}{x} = \int \frac{dx}{x}\int x^{\lambda}\,d\lambda,$$

1) Vide Commentationem **464** supra, p. 155, laudatam. A. G.

2) Vide exempli gratia ibidem p. 432. A. G.

quaeri tantum debet hoc integrale $\int x^\lambda d\lambda$ spectata quantitate x ut constante, ita ut sola λ sit variabilis. Est vero

$$\int x^\lambda d\lambda = \frac{x^\lambda}{lx} + C,$$

quemadmodum ex elementis calculi exponentialis liquet. Hic vero cardo rei in hoc versatur, ut istud integrale certa lege definiatur, quam deinceps etiam in altera parte observari oportet. Statuamus ergo talia integralia ita capi, ut evanescant posito $\lambda = 0$, eritque

$$\int x^\lambda d\lambda = \frac{x^\lambda - 1}{lx},$$

quo pacto pro sinistra parte habebimus

$$\int d\lambda \int x^\lambda \frac{dx}{x} = \int \frac{dx}{x} \cdot \frac{x^\lambda - 1}{lx}.$$

19. Pro parte autem dextra habebimus

$$\int \frac{d\lambda}{\lambda} (b^\lambda - a^\lambda),$$

qua formula eadem lege integrata, ut facto $\lambda = 0$ prodeat nihilum, hunc valorem more hic recepto repraesentare licebit

$$\int \frac{dy}{y} (b^y - a^y) \begin{bmatrix} \text{ab } & y = 0 \\ \text{ad } & y = \lambda \end{bmatrix}.$$

Hic enim nil aliud fecimus, nisi quod pro λ scripsimus y et facta integratione loco y eius valorem λ restitui assumsimus, sicque assecuti sumus sequentem formulam

$$\int (x^\lambda - 1) \frac{dx}{xlx} \begin{bmatrix} \text{ab } & x = a \\ \text{ad } & x = b \end{bmatrix} = \int \frac{dy}{y} (b^y - a^y) \begin{bmatrix} \text{ab } & y = 0 \\ \text{ad } & y = \lambda \end{bmatrix},$$

quam tanquam theorema utilissimum spectare licet.

20. Vi ergo huius theorematis nanciscimur sequentes reductiones

$$\int (x^n - 1) \frac{dx}{xlx} \begin{bmatrix} \text{ab } & x = a \\ \text{ad } & x = b \end{bmatrix} = \int \frac{dy}{y} (b^y - a^y) \begin{bmatrix} \text{ab } & y = 0 \\ \text{ad } & y = n \end{bmatrix}$$

et

$$\int (x^m - 1) \frac{dx}{xlx} \begin{bmatrix} \text{ab } & x = a \\ \text{ad } & x = b \end{bmatrix} = \int \frac{dy}{y} (b^y - a^y) \begin{bmatrix} \text{ab } & y = 0 \\ \text{ad } & y = m \end{bmatrix};$$

quare si formula posterior a priore subtrahatur, erit

$$\int (x^n - x^m)\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix} = \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=n\end{bmatrix} - \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=m\end{bmatrix};$$

verum ista formula ad dextram posita per reductionem in Lemmate 3 ostensam revocatur ad hanc formam simpliciorem

$$\int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=m\\ \text{ad } y=n\end{bmatrix};$$

unde patet hoc modo ipsum hoc insigne theorema etiam ex nostris principiis investigari potuisse.

21. Hoc autem theoremate generalissimo Vir ingeniosissimus est usus ad theorema meum demonstrandum, quo ostendi esse

$$\int (x^n - x^m)\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\,\frac{n}{m};$$

tantum enim opus erat, ut caperetur $a=0$ et $b=1$, quo pacto formula ad dextram posita integralis abit in

$$\int \frac{dy}{y}\begin{bmatrix}\text{ab } y=m\\ \text{ad } y=n\end{bmatrix},$$

cuius valor manifesto fit $ln - lm = l\,\frac{n}{m}$, quae est nova demonstratio mei theorematis, cuiusmodi quidem dudum [1]) plures alias dederam.

OBSERVATIONES IN THEOREMA D. DE LA GRANGE

$$\int \frac{(x^n - x^m)\,dx}{(1+x^r)\,lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = l\,\frac{\text{tang.}\,\dfrac{(n+1)\pi}{2r}}{\text{tang.}\,\dfrac{(m+1)\pi}{2r}}$$

22. Quia hic ambo exponentes m et n neque a se invicem neque ab exponente r pendent, manifestum est pro utraque potestate x^m et x^n seorsim integrale talem formam habere debere

$$\int \frac{x^n\,dx}{(1+x^r)\,lx} = l\,\text{tang.}\,\frac{(n+1)\pi}{2r} + C \quad \text{et} \quad \int \frac{x^m\,dx}{(1+x^r)\,lx} = l\,\text{tang.}\,\frac{(m+1)\pi}{2r} + C.$$

1) Vide Commentationem 464 supra, p. 155, laudatam. A. G.

Si enim posterior forma a priore subtrahatur, constans C ex calculo egreditur et ipsum integrale propositum resultat. Hic igitur plurimum intererit valorem istius constantis C determinasse.

23. Inter formulas integrales, quarum valores pro casu, quo post integrationem variabilis infinita statuitur, ex primis principiis calculi integralis assignavi[1]), reperitur ista

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{\pi}{2k \cos. \frac{n\pi}{2k}} = \frac{\pi}{2k \sin. \frac{(k+n)\pi}{2k}},$$

ubi autem assumitur exponentem n non maiorem capi quam k. Quodsi iam hic exponens n ut variabilis tractetur spectata ipsa x ut constante et utrinque per dn multiplicetur denuoque integretur, formula sinistra erit

$$\int dn \int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} = \int \frac{dx}{x(1+x^{2k})} \int x^{k+n} dn,$$

ubi postremum integrale fit

$$\int x^{k+n} dn = \frac{x^{k+n}}{lx} + C.$$

Ut autem hoc integrale determinetur, constantem ita definiamus, ut id evanescat posito $n=0$, unde obtinetur

$$\int x^{k+n} dn = \frac{x^{k+n} - x^k}{lx},$$

ita ut formula integralis ad sinistram posita futura sit

$$\int \frac{x^{k+n} - x^k}{1+x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix}.$$

24. Pro parte dextra autem habebimus hoc integrale

$$\int \frac{\pi dn}{2k \sin. \frac{(k+n)\pi}{2k}}$$

etiam ita sumendum, ut evanescat posito $n=0$. Hunc in finem statuamus

1) L. EULERI Commentatio 59 (indicis ENESTROEMIANI): *Theoremata circa reductionem formularum integralium ad quadraturam circuli*, Miscellanea Berolin. 7, 1743, p. 91; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 1, imprimis p. 29; vide porro *Institutionum calculi integralis* vol. I, § 351; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 225. A. G.

angulum $\frac{(k+n)\pi}{2k}=\varphi$, et quia hinc erit $d\varphi=\frac{\pi\, dn}{2k}$, formula nostra integranda erit $\int\frac{d\varphi}{\sin.\varphi}$, cuius integrale per regulas notas in genere est

$$l\,\mathrm{tang.}\,\frac{1}{2}\varphi+C=l\,\mathrm{tang.}\,\frac{(k+n)\pi}{4k}+C,$$

quod facto $n=0$ abit in $l\,\mathrm{tang.}\,\frac{\pi}{4}+C$. Quare cum $\mathrm{tang.}\,\frac{\pi}{4}=1$ et $l1=0$, evidens est constantem C fore $=0$, ita ut integrale hoc quaesitum sit $l\,\mathrm{tang.}\,\frac{(k+n)\pi}{4k}$. Hinc ergo assecuti sumus istam reductionem generalem

$$\int\frac{x^{k+n}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\mathrm{tang.}\,\frac{(k+n)\pi}{4k},$$

ubi autem probe notari oportet exponentes m et n maiores capi non licere quam k.

25. Cum igitur loco n alium numerum m sumendo simili modo sit

$$\int\frac{x^{k+m}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\mathrm{tang.}\,\frac{(k+m)\pi}{4k},$$

subtrahatur ista formula a praecedente et obtinebitur ista

$$\int\frac{x^{k+n}-x^{k+m}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\frac{\mathrm{tang.}\,\frac{(k+n)\pi}{4k}}{\mathrm{tang.}\,\frac{(k+m)\pi}{4k}},$$

quae manifesto cum forma proposita congruit, si modo loco $k+n-1$ scribatur n et m loco $k+m-1$, at loco exponentis $2k$ scribatur r; tum enim manifesto fiet

$$\int\frac{x^n-x^m}{1+x^r}\cdot\frac{dx}{lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\frac{\mathrm{tang.}\,\frac{(n+1)\pi}{2r}}{\mathrm{tang.}\,\frac{(m+1)\pi}{2r}}.$$

26. Quoniam ista analysis nos perduxit ad hanc formam

$$\int\frac{x^{k+n}-x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\mathrm{tang.}\,\frac{(k+n)\pi}{4k},$$

hic maximi momenti erit observasse semper fore

$$\int\frac{x^k}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=0,$$

id quod ita ostendere possum. Ponatur $x^k = z$; erit

$$x^{k-1}dx = \frac{dz}{k} \quad \text{et} \quad lx = \frac{lz}{k}$$

sicque ista formula induet hanc formam $\int \frac{dz}{(1+zz)lz}$, ubi termini integrationis etiamnunc sunt $z = 0$ et $z = \infty$. Fiat porro $z = \text{tang.}\, \varphi$, unde termini integrationis erunt $\varphi = 0$ et $\varphi = \frac{\pi}{2}$; hinc autem ob $d\varphi = \frac{dz}{1+zz}$ nascetur ista formula

$$\int \frac{d\varphi}{l \operatorname{tang} \varphi} \begin{bmatrix} \text{a} & \varphi = 0 \\ \text{ad} & \varphi = \frac{\pi}{2} \end{bmatrix},$$

cuius valorem in nihilum abire ostendi debet.

27. Ad hoc demonstrandum statuatur axis $IH = \frac{\pi}{2}$ (Fig. 2), super quo ab initio I sumta abscissa indefinita $Ip = \varphi$ applicata sit $= \frac{1}{l \operatorname{tang.} \varphi}$. Quodsi ergo hic axis IH in O bisecetur, ut sit $IO = \frac{\pi}{4}$, in hoc puncto applicata erit

$$= \frac{1}{l \operatorname{tang.} \frac{\pi}{4}} = \infty.$$

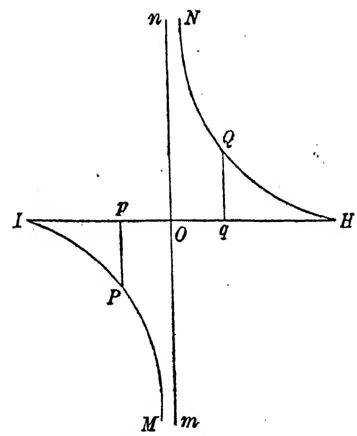

Fig. 2.

Iam ab hoc puncto O utrinque capiantur intervalla aequalia $Op = Oq = \omega$ et pro puncto p erit $\varphi = \frac{\pi}{4} - \omega$ sicque in hoc puncto p applicata erit

$$= \frac{1}{l \operatorname{tang.} \left(\frac{\pi}{4} - \omega\right)};$$

est vero $\operatorname{tang.} \left(\frac{\pi}{4} - \omega\right) = \operatorname{cot.} \left(\frac{\pi}{4} + \omega\right)$, quare, cum sit $l \operatorname{cot.} = - l \operatorname{tang.}$, applicata in hoc puncto p erit

$$= \frac{-1}{l \operatorname{tang.} \left(\frac{\pi}{4} + \omega\right)};$$

at quia est $Iq = \frac{\pi}{4} + \omega$, erit applicata in puncto q

$$= \frac{+1}{l \operatorname{tang.} \left(\frac{\pi}{4} + \omega\right)}$$

sicque aequalis est applicatae in p, sed in contrarium vergens. Ita si applicata sursum directa fuerit qQ, in puncto p eadem applicata deorsum erit directa $pP = qQ$.

28. Quodsi ergo talis curva super axe $IH = \frac{\pi}{2}$ exstruatur, ita ut abscissae φ respondeat applicata $\frac{1}{l \operatorname{tang.} \varphi}$, haec curva ex duabus portionibus inter se perfecte aequalibus constabit circa punctum medium O ita dispositis, ut curva sinistra sit IPM in infinitum descendens ad asymptotam Om, pars autem dextra simili modo a H sinistrorsum sursum ascendet ad asymptotam On. Quare cum formula integralis $\int \frac{d\varphi}{l \operatorname{tang.} \varphi}$ a $\varphi = 0$ ad $\varphi = \frac{\pi}{2}$ extensa exprimat totius huius curvae ab I usque ad H protensae aream, evidens est totam hanc aream ad nihilum redigi, quia portio eius negative sumenda perfecte similis est portioni positive sumendae.

29. Sic igitur per demonstrationem omnino singularem evictum est semper esse

$$\int \frac{x^k}{1+x^{2k}} \cdot \frac{dx}{x\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = 0,$$

quod certe est theorema in hoc genere maxime notatu dignum. Quodsi ergo cum illustri D. DE LA GRANGE statuamus $2k = r$, erit

$$\int \frac{x^{\frac{1}{2}r-1}\,dx}{(1+x^r)\,lx} = 0;$$

praeterea vero pro nostra formula § 24 exhibita ob

$$\int \frac{x^k\,dx}{(1+x^{2k})\,x\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = 0$$

deducitur istud theorema omnino notabile

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = l \operatorname{tang.} \frac{(k+n)\pi}{4k},$$

quod more D. DE LA GRANGE ita proponi potest

$$\int \frac{x^n\,dx}{(1+x^r)\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = l \operatorname{tang.} \frac{(n+1)\pi}{2r};$$

sicque patet constantem illam supra (§ 22) a nobis inductam revera nihilo aequari.

30. Quoniam demonstratio huius theorematis methodo satis insueta innititur, eius veritatem per series ostendisse iuvabit. Ad hoc autem valorem formulae

$$\int \frac{x^{\lambda-1}dx}{(1+x^r)lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix}$$

in duas partes divelli necesse est (scilicet loco n scribendo $\lambda-1$), quae sint

$$P=\int \frac{x^{\lambda-1}dx}{(1+x^r)lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} \quad \text{et} \quad Q=\int \frac{x^{\lambda-1}dx}{(1+x^r)lx} \begin{bmatrix} \text{ab } x=1 \\ \text{ad } x=\infty \end{bmatrix},$$

ita ut $P+Q$ exprimat valorem, quem quaerimus. Nunc in posteriore parte loco x scribamus $\frac{1}{z}$ fietque

$$Q=\int \frac{z^{-\lambda}}{1+z^{-r}}\cdot\frac{dz}{zlz} \begin{bmatrix} \text{a } z=1 \\ \text{ad } z=0 \end{bmatrix} = \int \frac{z^{r-\lambda}}{1+z^r}\cdot\frac{dz}{zlz} \begin{bmatrix} \text{a } z=1 \\ \text{ad } z=0 \end{bmatrix}$$

et commutatis terminis integrationis

$$Q=-\int \frac{z^{r-\lambda}}{1+z^r}\cdot\frac{dz}{zlz} \begin{bmatrix} \text{a } z=0 \\ \text{ad } z=1 \end{bmatrix}.$$

Nunc autem loco z scribamus x; quia termini integrationis utrinque sunt iidem, erit

$$P+Q=\int \frac{x^{\lambda}-x^{r-\lambda}}{1+x^r}\cdot\frac{dx}{xlx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix},$$

cuius ergo valor formulae propositae est aequalis.

31. Iam fractionem $\frac{1}{1+x^r}$ in seriem infinitam convertamus

$$1-x^r+x^{2r}-x^{3r}+x^{4r}-\text{etc.},$$

cuius singuli termini in $\frac{dx}{xlx}(x^{\lambda}-x^{r-\lambda})$ ducti producunt

$$\frac{dx}{xlx}(x^{\lambda}-x^{r-\lambda})-\frac{dx}{xlx}(x^{r+\lambda}-x^{2r-\lambda})+\frac{dx}{xlx}(x^{2r+\lambda}-x^{3r-\lambda})-\frac{dx}{xlx}(x^{3r+\lambda}-x^{4r-\lambda})+\text{etc.}$$

Cum autem per theorema principale in hoc genere sit

$$\int \frac{dx}{xlx}(x^{\alpha}-x^{\gamma}) \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l\frac{\alpha}{\beta},$$

singulis membris hoc modo integratis prodibit

$$P+Q = l\frac{\lambda}{r-\lambda} - l\frac{r+\lambda}{2r-\lambda} + l\frac{2r+\lambda}{3r-\lambda} - l\frac{3r+\lambda}{4r-\lambda} + \text{etc.}$$

32. Omnes hos logarithmos in unicum compingere licebit ratione habita signi cuiusque hocque modo reperietur fore

$$P+Q = l\frac{\lambda}{r-\lambda}\cdot\frac{2r-\lambda}{r+\lambda}\cdot\frac{2r+\lambda}{3r-\lambda}\cdot\frac{4r-\lambda}{3r+\lambda}\cdot\frac{4r+\lambda}{5r-\lambda}\cdot\frac{6r-\lambda}{5r+\lambda}\cdot\text{etc.}$$

At vero in *Introductione in Analysin Infinitorum*[1]) p. 147 ostendi esse

$$\text{tang.}\frac{m\pi}{2n} = \frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}\cdot\text{etc.},$$

quae series manifesto in inventam transformatur statuendo $m=\lambda$ et $n=r$, ita ut nunc sit $P+Q = l\,\text{tang.}\frac{\lambda\pi}{2r}$, prorsus uti supra est inventum.

ADDITAMENTUM

33. In dissertatione Actorum Tomo V, parte I, inserta[2]), unde desumsi hoc theorema

$$\int\frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = \frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

simul occurrunt sequentia

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = \frac{\pi}{k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix} = \frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k},$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{2k}\,\text{tang.}\frac{n\pi}{2k},$$

1) Vide *Introductionem in analysin infinitorum*, t. I cap. XI; LEONHARDI EULERI *Opera omnia*, series I, vol. 8. A. G.

2) Vide Commentationem 572 huius voluminis, imprimis § 23 et seq. A. G.

$$\int \frac{x^{k-n}+x^{k+n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{2\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int \frac{x^{k-n}+x^{k+n}}{1+2\,x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

$$\int \frac{x^{k\pm n}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

quas formulas ergo simili modo tractare operae pretium erit.

34. Incipiamus igitur a formula

$$\int \frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2k\cos.\frac{n\pi}{2k}},$$

quia praecedens cum formula iam tractata prorsus conveniret; quae si ducatur in dn et ita integretur, ut integrale evanescat posito $n=0$, quoniam est

$$\int x^{k-n}dn=-\frac{x^{k-n}-x^k}{lx} \quad \text{et} \quad \int x^{k+n}dn=\frac{x^{k+n}-x^k}{lx},$$

tum vero, ut ante vidimus,

$$\int \frac{\pi\,dn}{2k\cos.\frac{n\pi}{2k}}=l\,\text{tang.}\frac{(k+n)\pi}{4k},$$

prodibit haec integratio

$$\int \frac{x^{k+n}-x^{k-n}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\,\text{tang.}\frac{(k+n)\pi}{4k},$$

qui ergo valor prorsus convenit cum eo, quem pro formula

$$\int \frac{x^{k+n}}{1+x^{2k}}\cdot\frac{dx}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}$$

invenimus.

35. Simili modo tractemus sequentem formulam

$$\int \frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\cdot\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k},$$

quae ducta in dn et ut supra integrata praebet a parte sinistra

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix},$$

a parte autem dextra

$$\int \frac{\pi\, dn}{k} \operatorname{tang.} \frac{n\pi}{2k} = \int \frac{\pi\, dn \sin. \frac{n\pi}{2k}}{k \cos. \frac{n\pi}{2k}}.$$

Ad hoc integrandum fiat $\frac{n\pi}{2k} = \varphi$ eritque $\frac{\pi dn}{k} = 2d\varphi$ sicque formula integranda erit

$$2\int \frac{d\varphi \sin. \varphi}{\cos. \varphi} = -2l \cos. \varphi + C = -2l \cos. \frac{n\pi}{2k} + C.$$

Fiat igitur $n = 0$ esseque debebit $-2l1 + C = 0$ ideoque constans $C = 0$, quocirca haec integratio nobis suppeditat sequentem formulam

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1 - x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = -2l \cos. \frac{n\pi}{2k};$$

sequens autem formula $\begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = 1 \end{bmatrix}$ singulari evolutione non indiget, cum eius valor sit huius semissis.

36. Evolvamus casum, quo $k = 2$ et $n = 1$, et ex parte sinistra habemus

$$-\int \frac{(1-x)^2}{1-x^4} \cdot \frac{dx}{lx} = -\int \frac{1-x}{(1+x)(1+xx)} \cdot \frac{dx}{lx} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix};$$

at vero ex dextra parte $-2l \cos. \frac{\pi}{4} = 2l\sqrt{2} = l2$. Verum fractio $\frac{1-x}{(1+x)(1+xx)}$ resolvitur in has duas $\frac{1}{1+x} - \frac{x}{1+xx}$, unde formula nostra resolvitur in has duas

$$-\int \frac{dx}{(1+x)lx} + \int \frac{xdx}{(1+xx)lx} = l2.$$

Sed ex forma generali

$$\int \frac{x^{\lambda-1}dx}{(1+x^r)lx} = l \operatorname{tang.} \frac{\lambda\pi}{2r}$$

utriusque formulae valor in infinitum excrescit sicque nihil impedit, quominus differentia $= l2$.

37. Quodsi hic in posteriore formula statuamus $xx = z$, ea abibit in hanc $\int \frac{dz}{(1+z)lz}$, quae priori omnino est similis atque sub iisdem terminis integrationis continetur. Hic igitur iterum occurrit paradoxon prorsus simile illi, quod ab Illustr. DE LA GRANGE fuit memoratum; duae scilicet hic habentur formae prorsus pares $\int \frac{dx}{(1+x)lx}$ et $\int \frac{dz}{(1+z)lz}$, quarum utramque a termino 0 ad ∞ integrari oportet; nihilo tamen minus earum differentia non est nulla, sed, uti vidimus, $= l2$. Atque hinc solutio huius paradoxi in eo manifesto est sita, quod utriusque integralis valor in infinitum excrescit.

38. Quodsi binas postremas formulas eodem modo tractare et per dn multiplicatas integrare velimus, a parte sinistra resultat ista formula integralis

$$\int \frac{x^{k+n} - x^{k-n}}{1 + 2x^k \cos.\eta + x^{2k}} \cdot \frac{dx}{xlx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix};$$

pro dextra autem parte nanciscimur hanc formulam integralem

$$\int \frac{2\pi\, dn \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$$

a termino $n = 0$ extendendam. Verum haec integratio nullo modo succedit; si enim ponamus $\frac{n\pi}{k} = \varphi$, fiet $\frac{n\eta}{k} = \frac{\eta\varphi}{\pi} = \alpha\varphi$ ponendo $\frac{\eta}{\pi} = \alpha$, unde formula integranda erit $\frac{2}{\sin.\eta}\int \frac{d\varphi \sin.\alpha\varphi}{\sin.\varphi}$, cuius valor aliter nisi per signum summatorium exprimi non potest, sicque nulla concinna theoremata hinc derivare licet.

39. Quemadmodum autem hic exponentem n ut variabilem spectando transformationes per integrationem instituimus, ita etiam differentiatio egregias transformationes suppeditabit, quod argumentum unica formula principali illustrasse sufficiet. Consideremus scilicet hanc formulam

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{2k \cos.\frac{n\pi}{2k}},$$

quae sumto exponente n ut solo variabili continuo differentietur, ubi notandum est esse $d.x^{k+n} = x^{k+n}\, dnlx$. At vero pro formula $\frac{\pi}{2k\cos.\frac{n\pi}{2k}}$ scribamus lit-

teram ν, quae ergo spectanda erit tanquam functio ipsius n, cuius ergo differentialia cuiusque ordinis sunt in nostra potestate. Hinc igitur sequentes reductiones consequemur

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} lx = \frac{d\nu}{dn}$$

sive

$$\int \frac{x^{k+n-1}\,dx\,lx}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d\nu}{dn},$$

$$\int \frac{x^{k+n-1}\,dx\,(lx)^2}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{dd\nu}{dn^2},$$

$$\int \frac{x^{k+n-1}\,dx\,(lx)^3}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^3\nu}{dn^3},$$

$$\int \frac{x^{k+n-1}\,dx\,(lx)^4}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^4\nu}{dn^4},$$

$$\int \frac{x^{k+n-1}\,dx\,(lx)^5}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^5\nu}{dn^5}$$

etc.

40. Cum igitur hinc totum negotium ad differentialia continua ipsius ν reducatur, ea sequenti modo commodissime reperire licebit. Cum enim sit

$$\nu = \frac{\pi}{2k \cos. \frac{n\pi}{2k}},$$

erit $\nu \cos. \frac{n\pi}{2k} = \frac{\pi}{2k}$ hincque continuo differentiando obtinebimus sequentes formulas

$$\frac{d\nu}{dn} \cos. \frac{n\pi}{2k} - \frac{\pi}{2k}\nu \sin. \frac{n\pi}{2k} = 0,$$

$$\frac{dd\nu}{dn^2} \cos. \frac{n\pi}{2k} - \frac{2\pi}{2k}\frac{d\nu}{dn} \sin. \frac{n\pi}{2k} - \frac{\pi\pi}{4kk}\nu \cos. \frac{n\pi}{2k} = 0,$$

$$\frac{d^3\nu}{dn^3} \cos. \frac{n\pi}{2k} - \frac{3\pi}{2k}\frac{dd\nu}{dn^2} \sin. \frac{n\pi}{2k} - \frac{3\pi\pi}{4kk}\frac{d\nu}{dn} \cos. \frac{n\pi}{2k} + \frac{\pi^3}{8k^3}\nu \sin. \frac{n\pi}{2k} = 0,$$

$$\frac{d^4\nu}{dn^4} \cos. \frac{n\pi}{2k} - \frac{4\pi}{2k}\frac{d^3\nu}{dn^3} \sin. \frac{n\pi}{2k} - \frac{6\pi\pi}{4kk}\frac{dd\nu}{dn^2} \cos. \frac{n\pi}{2k} + \frac{4\pi^3}{8k^3}\frac{d\nu}{dn} \sin. \frac{n\pi}{2k} + \frac{\pi^4}{16k^4}\nu \cos. \frac{n\pi}{2k} = 0$$

etc.,

unde singula differentialia altiora ex inferioribus formari possunt.

41. Quo autem hae operationes magis subleventur, statuamus brevitatis gratia $\frac{\pi}{2k}=\alpha$, ut sit $\nu=\frac{\alpha}{\cos.\alpha n}$, atque singula differentialia ex superioribus aequationibus sequenti modo determinabuntur

$$\frac{d\nu}{dn}=\alpha\nu\,\mathrm{tang.}\,\alpha n,$$

$$\frac{dd\nu}{dn^2}=2\alpha\frac{d\nu}{dn}\,\mathrm{tang.}\,\alpha n+\alpha\alpha\nu,$$

$$\frac{d^3\nu}{dn^3}=3\alpha\frac{dd\nu}{dn^2}\,\mathrm{tang.}\,\alpha n+3\alpha\alpha\frac{d\nu}{dn}-\alpha^3\nu\,\mathrm{tang.}\,\alpha n,$$

$$\frac{d^4\nu}{dn^4}=4\alpha\frac{d^3\nu}{dn^3}\,\mathrm{tang.}\,\alpha n+6\alpha\alpha\frac{dd\nu}{dn^2}-4\alpha^3\frac{d\nu}{dn}\,\mathrm{tang.}\,\alpha n-\alpha^4\nu,$$

$$\frac{d^5\nu}{dn^5}=5\alpha\frac{d^4\nu}{dn^4}\,\mathrm{tang.}\,\alpha n+10\alpha\alpha\frac{d^3\nu}{dn^3}-10\alpha^3\frac{dd\nu}{dn^2}\,\mathrm{tang.}\,\alpha n-5\alpha^4\frac{d\nu}{dn}+\alpha^5\nu\,\mathrm{tang.}\,\alpha n$$

etc.

Quodsi brevitatis gratia insuper statuamus $\mathrm{tang.}\,\alpha n=t$ et praecedentes valores in sequentibus substituamus, reperiemus

$$\frac{d\nu}{dn}=\alpha\nu t,$$

$$\frac{dd\nu}{dn^2}=\alpha\alpha\nu(2tt+1),$$

$$\frac{d^3\nu}{dn^3}=\alpha^3\nu(6t^3+5t),$$

$$\frac{d^4\nu}{dn^4}=\alpha^4\nu(24t^4+28tt+5),$$

$$\frac{d^5\nu}{dn^5}=\alpha^5\nu(120t^5+180t^3+61t),$$

$$\frac{d^6\nu}{dn^6}=\alpha^6\nu(720t^6+1320t^4+662tt+61)$$

etc.

42. Ex consideratione harum expressionum facilis erui potest operatio, cuius ope ex qualibet earum expressionum sequens colligi potest. Sit enim pro differentiali ordinis indefiniti

$$\frac{d^\lambda\nu}{dn^\lambda}=\alpha^\lambda\nu P,$$

at pro ordine sequente

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu Q,$$

et quoniam vidimus valorem ipsius P talem habere formam

$$P = At^{\lambda} + Bt^{\lambda-2} + Ct^{\lambda-4} + Dt^{\lambda-6} + \text{etc.},$$

tum valor ipsius Q ex sequentibus binis seriebus erit compositus

$$\begin{aligned} Q = (\lambda+1)At^{\lambda+1} + (\lambda-1)Bt^{\lambda-1} + (\lambda-3)Ct^{\lambda-3} + (\lambda-5)Dt^{\lambda-5} + \text{etc.} \\ + \lambda At^{\lambda-1} + (\lambda-2)Bt^{\lambda-3} + (\lambda-4)Ct^{\lambda-5} + \text{etc.}, \end{aligned}$$

unde patet hanc determinationem ita repraesentari posse, ut sit

$$Q = \frac{td.Pt}{dt} + \frac{dP}{dt}.$$

43. Haec vero formula, qua ex cognito valore P sequens Q derivatur, etiam ex ipsa natura rei sequenti modo ostendi potest. Cum per hypothesin sit

$$\frac{d^{\lambda}\nu}{dn^{\lambda}} = \alpha^{\lambda}\nu P,$$

erit differentiando

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda}} = \alpha^{\lambda} Pd\nu + \alpha^{\lambda}\nu dP;$$

initio autem vidimus esse $\frac{d\nu}{dn} = \alpha\nu t$ sive $d\nu = \alpha\nu t dn$, quo valore substituto fit

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu Pt + \alpha^{\lambda}\nu\frac{dP}{dn};$$

tum vero assumsimus $t = \text{tang.}\,\alpha n$, unde differentiando fit $\alpha dn = \frac{dt}{1+tt}$, quo valore in postremo termino substituto obtinebitur

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu Pt + \alpha^{\lambda+1}\nu\frac{dP(1+tt)}{dt} = \alpha^{\lambda+1}\nu\left(Pt + \frac{dP(1+tt)}{dt}\right),$$

quae forma manifesto reducitur ad hanc

$$\frac{d^{\lambda+1}\nu}{dn^{\lambda+1}} = \alpha^{\lambda+1}\nu\,\frac{td.Pt + dP}{dt},$$

ita ut sit

$$Q = \frac{td.Pt + dP}{dt} = Pt + \frac{dP(1+tt)}{dt};$$

unde intelligitur, si sumatur $tt + 1 = 0$, quo facto in nostris formulis signa terminorum alternabuntur, et omissa littera t fieri $Q = P$; unde patet hoc casu omnes formulas superiores eundem valorem esse adepturas, id quod etiam ex formulis supra exhibitis manifestum est, ex quibus erit $2 - 1 = 1$, $6 - 5 = 1$, $24 - 28 + 5 = 1$, $120 - 180 + 61 = 1$, $720 - 1320 + 662 - 61 = 1$ etc., unde insigne criterium obtinetur, utrum formulae istae recte sint per calculum definitae.

INVESTIGATIO FORMULAE INTEGRALIS

$$\int \frac{x^{m-1}dx}{(1+x^k)^n}$$

CASU QUO POST INTEGRATIONEM STATUITUR $x=\infty$

Commentatio 588 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 42—54

1. Iam satis notum est huius formulae integrale [casu, quo $n=1$] partim logarithmos, partim arcus circulares complecti et partes logarithmicas hanc progressionem constituere

$$\begin{aligned}
&-\frac{2}{k}\cos.\frac{m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{k}+xx\right)}\\
&-\frac{2}{k}\cos.\frac{3m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{3\pi}{k}+xx\right)}\\
&-\frac{2}{k}\cos.\frac{5m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{5\pi}{k}+xx\right)}\\
&-\frac{2}{k}\cos.\frac{7m\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{7\pi}{k}+xx\right)}\\
&\quad\vdots\\
&-\frac{2}{k}\cos.\frac{im\pi}{k}\, l\sqrt{\left(1-2x\cos.\frac{i\pi}{k}+xx\right)},
\end{aligned}$$

ubi i denotat numerum imparem non maiorem quam k. Hinc si k fuerit numerus par, erit $i=k-1$; ac si k fuerit numerus impar, hanc progressionem continuari oportet usque ad $i=k$, eius vero coefficiens duplo minor capi debet seu loco $-\frac{2}{k}$ tantum scribi debet $-\frac{1}{k}$, cuius irregularitatis ratio in *Calculo Integrali*[1]) est exposita.

1) Vide notam p. 122. A. G.

2. Cum hae partes sponte iam evanescant posito $x=0$, statuamus statim $x=\infty$, et cum in genere sit

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega,$$

erit

$$l\sqrt{(1-2x\cos.\omega+xx)}=l(x-\cos.\omega)=lx-\frac{\cos.\omega}{x}=lx$$

ob $\frac{\cos.\omega}{x}=0$; omnes ergo illi logarithmi reducuntur ad eandem formam lx, quae multiplicanda est per hanc seriem

$$-\frac{2}{k}\cos.\frac{m\pi}{k}-\frac{2}{k}\cos.\frac{3m\pi}{k}-\frac{2}{k}\cos.\frac{5m\pi}{k}-\cdots-\frac{2}{k}\cos.\frac{im\pi}{k},$$

ubi, ut diximus, i denotat maximum numerum imparem ipso k non maiorem, hac tamen restrictione, ut, si k fuerit impar ideoque $i=k$, ultimum membrum ad dimidium reduci debeat. Quamobrem si huius progressionis summam investigare velimus, duo casus erunt constituendi, alter, quo k est numerus par et $i=k-1$, alter vero, quo k est impar et $i=k$.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i=k-1$

3. Hoc ergo casu posito $x=\infty$ formula $-\frac{2}{k}lx$ multiplicatur per hanc cosinuum seriem

$$\cos.\frac{m\pi}{k}+\cos.\frac{3m\pi}{k}+\cos.\frac{5m\pi}{k}+\cos.\frac{7m\pi}{k}+\cdots+\cos.\frac{(k-1)m\pi}{k},$$

cuius summam statuamus $=S$. Ducamus hanc seriem in $\sin.\frac{m\pi}{k}$, et cum in genere sit

$$\sin.\frac{m\pi}{k}\cos.\frac{im\pi}{k}=\frac{1}{2}\sin.\frac{(i+1)m\pi}{k}-\frac{1}{2}\sin.\frac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$S\sin.\frac{m\pi}{k}$$

$$=\frac{1}{2}\sin.\frac{2m\pi}{k}+\frac{1}{2}\sin.\frac{4m\pi}{k}+\frac{1}{2}\sin.\frac{6m\pi}{k}+\cdots+\frac{1}{2}\sin.\frac{(k-2)m\pi}{k}+\frac{1}{2}\sin.\frac{km\pi}{k}$$

$$-\frac{1}{2}\sin.\frac{2m\pi}{k}-\frac{1}{2}\sin.\frac{4m\pi}{k}-\frac{1}{2}\sin.\frac{6m\pi}{k}-\cdots-\frac{1}{2}\sin.\frac{(k-2)m\pi}{k},$$

ubi omnes termini praeter ultimum manifesto se destruunt, ita ut sit

$$S \sin. \frac{m\pi}{k} = \frac{1}{2} \sin. m\pi.$$

Iam vero, quia nostri coefficientes m et k supponuntur integri, utique erit $\sin. m\pi = 0$ ideoque etiam $S = 0$, nisi forte etiam fuerit $\sin. \frac{m\pi}{k} = 0$, qui autem casus locum habere nequit, quoniam in integratione formulae propositae $\frac{x^{m-1}dx}{(1+x^k)^n}$ semper assumi solet esse $m < k$. Hoc igitur modo evictum est casu, quo post integrationem statuitur $x = \infty$, omnes partes logarithmicas integralis se destruere.

EVOLUTIO CASUS ALTERIUS QUO EST k NUMERUS IMPAR ET $i = k$

4. Hoc ergo casu sumto $x = \infty$ formula lx multiplicatur per hanc seriem

$$-\frac{2}{k} \cos. \frac{m\pi}{k} - \frac{2}{k} \cos. \frac{3m\pi}{k} - \frac{2}{k} \cos. \frac{5m\pi}{k} - \cdots - \frac{1}{k} \cos. \frac{km\pi}{k},$$

ubi terminus penultimus est $-\frac{2}{k} \cos. \frac{(k-2)m\pi}{k}$, pro ultimo vero termino erit $\cos. m\pi = \pm 1$ signo superiore valente, si n sit numerus par, inferiore, si impar; quare remoto termino ultimo pro reliquis ponamus

$$\cos. \frac{m\pi}{k} + \cos. \frac{3m\pi}{k} + \cos. \frac{5m\pi}{k} + \cdots + \cos. \frac{(k-2)m\pi}{k} = S,$$

ita ut multiplicator ipsius logarithmi x sit

$$-\frac{2S}{k} - \frac{1}{k} \cos. m\pi.$$

Hinc procedendo ut ante fiet

$$S \sin. \frac{m\pi}{k}$$

$$= \frac{1}{2} \sin. \frac{2m\pi}{k} + \frac{1}{2} \sin. \frac{4m\pi}{k} + \frac{1}{2} \sin. \frac{6m\pi}{k} + \cdots + \frac{1}{2} \sin. \frac{(k-3)m\pi}{k} + \frac{1}{2} \sin. \frac{(k-1)m\pi}{k}$$
$$- \frac{1}{2} \sin. \frac{2m\pi}{k} - \frac{1}{2} \sin. \frac{4m\pi}{k} - \frac{1}{2} \sin. \frac{6m\pi}{k} - \cdots - \frac{1}{2} \sin. \frac{(k-3)m\pi}{k},$$

ubi iterum omnes termini praeter ultimum se mutuo tollunt, ita ut hinc prodeat

$$S \sin. \frac{m\pi}{k} = \frac{1}{2} \sin. \frac{(k-1)m\pi}{k} = \frac{1}{2} \sin. \left(m\pi - \frac{m\pi}{k}\right);$$

at vero est

$$\sin.\left(m\pi-\frac{m\pi}{k}\right)=\sin.m\pi\cos.\frac{m\pi}{k}-\cos.m\pi\sin.\frac{m\pi}{k},$$

ubi notetur esse $\sin.m\pi=0$ ob m numerum integrum; habebimus ergo

$$S\sin.\frac{m\pi}{k}=-\frac{1}{2}\cos.m\pi\sin.\frac{m\pi}{k}\quad\text{sive}\quad S=-\frac{1}{2}\cos.m\pi,$$

consequenter multiplicator ipsius lx erit

$$=\frac{1}{k}\cos.m\pi-\frac{1}{k}\cos.m\pi=0$$

sicque manifestum est, sive k sit numerus par sive impar, omnia membra logarithmica in nostro integrali se mutuo destruere, siquidem post integrationem statuamus $x=\infty$, quemadmodum hic semper supponimus.

5. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componitur. Hae autem partes sequentem progressionem constituere sunt compertae:

$$\frac{2}{k}\sin.\frac{m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}+\frac{2}{k}\sin.\frac{3m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}$$

$$+\frac{2}{k}\sin.\frac{5m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}}+\frac{2}{k}\sin.\frac{7m\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{7\pi}{k}}{1-x\cos.\frac{7\pi}{k}}$$

$$+\cdots+\frac{2}{k}\sin.\frac{im\pi}{k}\,\text{A tang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}},$$

ubi in ultimo membro est vel $i=k-1$ vel $i=k$; prius scilicet valet, si i est numerus par, posterius, si impar.

6. Cum etiam omnia haec membra evanescant posito $x=0$, faciamus pro instituto nostro $x=\infty$. In genere igitur fiet

$$\text{A tang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}=\text{A tang.}\left(-\text{tang.}\frac{i\pi}{k}\right).$$

Est vero

$$-\operatorname{tang.} \frac{i\pi}{k} = + \operatorname{tang.} \frac{(k-i)\pi}{k},$$

ex quo hic arcus fit $= \frac{(k-i)\pi}{k}$. Hinc ergo loco i scribendo successive numeros 1, 3, 5, 7 etc. istae partes nostri integralis quaesiti erunt

$$\frac{2(k-1)\pi}{kk} \sin. \frac{m\pi}{k} + \frac{2(k-3)\pi}{kk} \sin. \frac{3m\pi}{k} + \frac{2(k-5)\pi}{kk} \sin. \frac{5m\pi}{k} + \frac{2(k-7)\pi}{kk} \sin. \frac{7m\pi}{k}$$
$$+ \frac{2(k-9)\pi}{kk} \sin. \frac{9m\pi}{k} + \cdots + \frac{2(k-i)\pi}{kk} \sin. \frac{im\pi}{k},$$

ubi casu, quo k est numerus par, progredi oportet usque ad $i = k-1$, ac si k sit numerus impar, usque ad $i = k$.

7. Statuamus brevitatis gratia

$$(k-1) \sin. \frac{m\pi}{k} + (k-3) \sin. \frac{3m\pi}{k} + (k-5) \sin. \frac{5m\pi}{k} + \cdots + (k-i) \sin. \frac{im\pi}{k} = S,$$

ita ut integrale quaesitum sit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae se mutuo destruxerunt. Multiplicemus nunc utrinque per $2 \sin. \frac{m\pi}{k}$, et cum in genere sit

$$2 \sin. \frac{m\pi}{k} \sin. \frac{im\pi}{k} = \cos. \frac{(i-1)m\pi}{k} - \cos. \frac{(i+1)m\pi}{k},$$

facta substitutione erit

$$2S \sin. \frac{m\pi}{k} = (k-1) \cos. \frac{0m\pi}{k}$$
$$+ (k-3) \cos. \frac{2m\pi}{k} + (k-5) \cos. \frac{4m\pi}{k} + \cdots + \quad (k-i) \cos. \frac{(i-1)m\pi}{k}$$
$$-(k-1) \cos. \frac{2m\pi}{k} - (k-3) \cos. \frac{4m\pi}{k} - \cdots - (k-i+2) \cos. \frac{(i-1)m\pi}{k} - (k-i) \cos. \frac{(i+1)m\pi}{k},$$

quae series manifesto contrahitur in sequentem

$$2S \sin. \frac{m\pi}{k} = k - 1 - 2\cos. \frac{2m\pi}{k} - 2\cos. \frac{4m\pi}{k} - 2\cos. \frac{6m\pi}{k} - \cdots - 2\cos. \frac{(i-1)m\pi}{k}$$
$$- (k-i) \cos. \frac{(i+1)m\pi}{k},$$

ubi primo et ultimo membro sublatis regularem termini intermedii constituunt seriem, pro cuius valore investigando ponamus

$$T=\cos.\frac{2m\pi}{k}+\cos.\frac{4m\pi}{k}+\cos.\frac{6m\pi}{k}+\cdots+\cos.\frac{(i-1)m\pi}{k},$$

ita ut sit

$$2S\sin.\frac{m\pi}{k}=k-1-2T-(k-i)\cos.\frac{(i+1)m\pi}{k}.$$

Hic autem iterum convenit duos casus perpendere, prout k fuerit par vel impar.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i=k-1$

8. Hoc ergo casu habebimus

$$T=\cos.\frac{2m\pi}{k}+\cos.\frac{4m\pi}{k}+\cos.\frac{6m\pi}{k}+\cdots+\cos.\frac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2\sin.\frac{m\pi}{k}$ et per reductiones supra indicatas habebimus

$$2T\sin.\frac{m\pi}{k}=\sin.\frac{3m\pi}{k}+\sin.\frac{5m\pi}{k}+\cdots+\sin.\frac{(k-3)m\pi}{k}+\sin.\frac{(k-1)m\pi}{k}$$
$$-\sin.\frac{m\pi}{k}-\sin.\frac{3m\pi}{k}-\sin.\frac{5m\pi}{k}-\cdots-\sin.\frac{(k-3)m\pi}{k};$$

deletis igitur terminis se mutuo tollentibus erit

$$2T\sin.\frac{m\pi}{k}=-\sin.\frac{m\pi}{k}+\sin.\frac{(k-1)m\pi}{k}.$$

Est vero

$$\sin.\frac{(k-1)m\pi}{k}=\sin.\left(m\pi-\frac{m\pi}{k}\right)=\sin.m\pi\cos.\frac{m\pi}{k}-\cos.m\pi\sin.\frac{m\pi}{k},$$

ubi $\sin.m\pi=0$, quamobrem fiet

$$2T=-1-\cos.m\pi.$$

9. Invento valore pro T colligitur fore

$$2S\sin.\frac{m\pi}{k}=k \quad \text{ideoque} \quad S=\frac{k}{2\sin.\frac{m\pi}{k}}.$$

Denique vero ipse valor formulae nostrae integralis, quem quaerimus, erit $\frac{2\pi S}{kk}$ et nunc manifestum est integrale nostrae formulae casu, quo S est numerus par, fore $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, siquidem post integrationem statuatur $x=\infty$.

EVOLUTIO ALTERIUS CASUS QUO k EST NUMERUS IMPAR ET $i=k$

10. Hoc ergo casu est

$$T=\cos.\frac{2m\pi}{k}+\cos.\frac{4m\pi}{k}+\cos.\frac{6m\pi}{k}+\cdots+\cos.\frac{(k-1)m\pi}{k},$$

quae series multiplicata per $2\sin.\frac{m\pi}{k}$ producet ut ante

$$2T\sin.\frac{m\pi}{k}=\sin.\frac{3m\pi}{k}+\sin.\frac{5m\pi}{k}+\cdots+\sin.\frac{(k-2)m\pi}{k}+\sin.\frac{km\pi}{k}$$
$$-\sin.\frac{m\pi}{k}-\sin.\frac{3m\pi}{k}-\sin.\frac{5m\pi}{k}-\cdots-\sin.\frac{(k-2)m\pi}{k},$$

unde deletis terminis se mutuo tollentibus reperietur

$$2T\sin.\frac{m\pi}{k}=-\sin.\frac{m\pi}{k}+\sin.m\pi$$

ideoque

$$2T=-1+\frac{\sin.m\pi}{\sin.\frac{m\pi}{k}}=-1$$

ob $\sin.m\pi=0$, hincque porro fiet

$$2S\sin.\frac{m\pi}{k}=k;$$

quare cum valor integralis quaesitus sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti praecedente casu. Hinc ergo deducimus sequens

THEOREMA

11. *Si haec formula differentialis*

$$\frac{x^{m-1}dx}{1+x^k}$$

ita integretur, ut posito $x=0$ *integrale evanescat, tum vero statuatur* $x=\infty$,

valor inde resultans semper erit

$$\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

sive k sit numerus par sive impar.

Huius theorematis demonstratio ex praecedentibus est manifesta.

12. In evolutione huius formulae assumsimus esse $m<k$, quia alioquin membra logarithmica se non destruissent; at vero ne hac quidem limitatione nunc amplius est opus. Casu enim, quo foret $m=k$, integrale formulae $\frac{x^{m-1}dx}{1+x^k}$ esset $\frac{1}{k}l(1+x^k)$, quod facto $x=\infty$ fieret etiam ∞; verum hoc idem indicat nostrum integrale esse $\frac{\pi}{k\sin.\pi}=\infty$. Dummodo ergo m non fuerit maius quam k, nostra formula veritati semper est consentanea.

13. Quin etiam ne quidem necesse est, ut exponentes m et k sint numeri integri, dummodo non fuerit $m>k$; si enim fuerit $m=\frac{\mu}{\lambda}$ et $k=\frac{\varkappa}{\lambda}$, erit valor per nostram formulam $\frac{\lambda\pi}{\varkappa\sin.\frac{\mu\pi}{\varkappa}}$, cuius veritas ita ostenditur. Quia hoc casu formula integranda est

$$\int\frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\varkappa}{\lambda}}}\cdot\frac{dx}{x},$$

statuatur $x=y^\lambda$; erit $\frac{dx}{x}=\frac{\lambda dy}{y}$ et formula fiet

$$\int\frac{y^\mu}{1+y^\varkappa}\cdot\frac{\lambda dy}{y}=\lambda\int\frac{y^{\mu-1}dy}{1+y^\varkappa},$$

cuius valor utique erit $\frac{\lambda\pi}{\varkappa\sin.\frac{\mu\pi}{\varkappa}}$.

ALIA DEMONSTRATIO THEOREMATIS

14. Denotet P valorem integralis $\int\frac{x^m}{1+x^k}\cdot\frac{dx}{x}$ a termino $x=0$ usque ad $x=1$, at Q valorem eiusdem integralis a termino $x=1$ usque ad $x=\infty$, ita ut $P+Q$ praebeat eum ipsum valorem, qui in theoremate continetur. Nunc pro valore Q inveniendo statuatur $x=\frac{1}{y}$, unde fit $\frac{dx}{x}=-\frac{dy}{y}$, fietque

$$Q = \int \frac{y^{-m}}{1+y^{-k}} \cdot \frac{-dy}{y} = -\int \frac{y^{k-m}}{1+y^k} \cdot \frac{dy}{y}$$

a termino $y = 1$ usque ad $y = 0$. Hinc igitur commutatis terminis erit

$$Q = +\int \frac{y^{k-m}}{1+y^k} \cdot \frac{dy}{y}$$

a termino $y = 0$ usque ad $y = 1$. Iam quia hoc integrali expedito littera y ex calculo egreditur, loco y scribere licebit x, ita ut sit

$$Q = \int \frac{x^{k-m}}{1+x^k} \cdot \frac{dx}{x},$$

quo facto habebimus

$$P + Q = \int \frac{x^m + x^{k-m}}{1+x^k} \cdot \frac{dx}{x}$$

a termino $x = 0$ usque ad terminum $x = 1$. Verum non ita pridem demonstravi[1]) valorem huius formulae integralis intra terminos $x = 0$ et $x = 1$ contentum esse $= \frac{\pi}{k \sin. \frac{m\pi}{k}}$. Hinc igitur nascitur sequens theorema non minus notatu dignum.

THEOREMA

15. *Valor huius formulae integralis*

$$\int \frac{x^m + x^{k-m}}{1+x^k} \cdot \frac{dx}{x}$$

intra terminos $x = 0$ et $x = 1$ contentus aequalis est valori istius integralis

$$\int \frac{x^m}{1+x^k} \cdot \frac{dx}{x}$$

intra terminos $x = 0$ et $x = \infty$ contento.

16. His expensis formulam integralem in titulo propositam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in subsidium vocemus

1) Vide praeter § 27 Commentationis praecedentis etiam Commentationem 463 (indicis Enestroemiani): *De valore formulae integralis* $\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z} (lz)^\mu$ *casu, quo post integrationem ponitur* $z = 1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 30; Leonhardi Euleri *Opera omnia*, series I, vol. 17, p. 384, imprimis p. 388. A. G.

sequentem reductionem

$$\int \frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \frac{Ax^m}{(1+x^k)^\lambda} + B\int \frac{x^{m-1}dx}{(1+x^k)^\lambda},$$

unde facta differentiatione prodit sequens aequatio

$$\frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \frac{mAx^{m-1}dx}{(1+x^k)^\lambda} - \frac{\lambda kAx^{m+k-1}dx}{(1+x^k)^{\lambda+1}} + \frac{Bx^{m-1}dx}{(1+x^k)^\lambda},$$

quae aequatio per $x^{m-1}dx$ divisa ac per $(1+x^k)^\lambda$ multiplicata terminum negativum a dextra ad sinistram transponendo erit

$$\frac{1+\lambda kAx^k}{1+x^k} = mA + B,$$

quae aequatio manifesto subsistere nequit, nisi sit $\lambda kA = 1$ sive $A = \frac{1}{\lambda k}$, unde erit $1 = mA + B = \frac{m}{\lambda k} + B$, sicque erit $B = 1 - \frac{m}{\lambda k}$.

17. Inventis his valoribus pro litteris A et B primum assumimus integralia ita capi, ut evanescant posito $x = 0$; tum vero posito $x = \infty$, quia exponens n minor supponitur quam k, membrum absolutum littera A affectum sponte evanescit, ita ut hoc casu $x = \infty$ fiat

$$\int \frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \left(1 - \frac{m}{\lambda k}\right)\int \frac{x^{m-1}dx}{(1+x^k)^\lambda}.$$

Quodsi iam primo capiamus $\lambda = 1$, quia ante invenimus pro eodem casu $x = \infty$ esse

$$\int \frac{x^{m-1}dx}{1+x^k} = \frac{\pi}{k \sin. \frac{m\pi}{k}},$$

habebimus valorem istius integralis

$$\int \frac{x^{m-1}dx}{(1+x^k)^2} = \left(1 - \frac{m}{k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}},$$

siquidem integrale etiam a termino $x = 0$ usque ad terminum $x = \infty$ extendatur.

18. Quodsi iam simili modo ponamus $\lambda = 2$, reperietur pro iisdem terminis integrationis

$$\int \frac{x^{m-1}dx}{(1+x^k)^3} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}};$$

eodem modo si litterae λ continuo maiores valores tribuantur, reperientur sequentes integralium formae omni attentione dignae

$$\int \frac{x^{m-1}dx}{(1+x^k)^4} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}dx}{(1+x^k)^5} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}dx}{(1+x^k)^6} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$$

etc.

19. Quare si littera n denotet numerum quemcunque integrum pro formula in titulo expressa, si eius integrale a termino $x = 0$ usque ad $x = \infty$ extendatur, eius valor sequenti modo se habebit:

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\cdots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

qui ergo conveniet huic formulae integrali

$$\int \frac{x^{m-1}dx}{(1+x^k)^n}.$$

20. Hic quidem necessario pro n alii numeri praeter integros accipi non licet; at vero per methodum interpolationum, quae fusius iam passim est explicata[1]), hanc integrationem etiam ad casus, quibus exponens n est numerus fractus, extendere licet. Quodsi enim sequentes formulae integrales a ter-

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1) Vide Commentationem 254 (indicis ENESTROEMIANI): *De expressione integralium per factores*, Novi comment. acad. sc. Petrop. 6 (1756/7), 1761, p. 115; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 233. A. G.

mino $y=0$ usque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit

$$\int\frac{x^{m-1}dx}{(1+x^k)^n}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\int y^{nk-m-1}dy(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}dy(1-y^k)^{\frac{m}{k}-1}}.$$

Unde, si fuerit $m=1$ et $k=2$, sequitur fore

$$\int\frac{dx}{(1+xx)^n}=\frac{\pi}{2}\int\frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}:\int\frac{dy}{\sqrt{(1-yy)}}=\int\frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}.$$

Ita, si $n=\frac{3}{2}$, erit

$$\int\frac{dx}{(1+xx)^{\frac{3}{2}}}=\int\frac{ydy}{\sqrt{(1-yy)}},$$

cuius veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{(1+xx)}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae facto $x=\infty$ et $y=1$ utique fiunt aequalia. Caeterum pro hac integratione generali notasse iuvabit exponentem unitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

INVESTIGATIO VALORIS INTEGRALIS

$$\int \frac{x^{m-1}\,dx}{1-2x^k \cos.\theta + x^{2k}}$$

A TERMINO $x=0$ USQUE AD $x=\infty$ EXTENSI

Commentatio 589 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 55—75

1. Quaeramus primo integrale formulae propositae indefinitum atque adeo omnes operationes ex primis Analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolvi nequit, sit in genere eius factor duplicatus quicunque $1-2x\cos.\omega+xx$; evidens enim est denominatorem fore productum ex k huiusmodi factoribus duplicatis. Cum igitur posito hoc factore $=0$ fiat $x=\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega$, etiam ipse denominator duplici modo evanescere debebit, sive si ponatur

$$x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega \quad \text{sive} \quad x=\cos.\omega-\sqrt{-1}\cdot\sin.\omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, ut sit

$$(\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega)^\lambda=\cos.\lambda\omega\pm\sqrt{-1}\cdot\sin.\lambda\omega;$$

hinc igitur erit

$$x^k=\cos.k\omega\pm\sqrt{-1}\cdot\sin.k\omega \quad \text{et} \quad x^{2k}=\cos.2k\omega\pm\sqrt{-1}\cdot\sin.2k\omega.$$

Substituamus ergo hos valores et denominator noster evadet

$$1-2\cos.\theta\cos.k\omega+\cos.2k\omega\mp2\sqrt{-1}\cdot\cos.\theta\sin.k\omega\pm\sqrt{-1}\cdot\sin.2k\omega.$$

2. Perspicuum igitur est huius aequationis tam terminos reales quam imaginarios seorsim se mutuo tollere debere, unde nascuntur hae duae aequationes

$$\text{I.}\quad 1-2\cos.\theta\cos.k\omega+\cos.2k\omega=0,$$

$$\text{II.}\quad -2\cos.\theta\sin.k\omega+\sin.2k\omega=0.$$

Cum igitur sit

$$\sin.2k\omega=2\sin.k\omega\cos.k\omega,$$

posterior aequatio induet hanc formam

$$-2\cos.\theta\sin.k\omega+2\sin.k\omega\cos.k\omega=0,$$

quae per $2\sin.k\omega$ divisa dat

$$\cos.k\omega=\cos.\theta$$

ideoque

$$\cos.2k\omega=\cos.2\theta=\cos.\theta^2-\sin.\theta^2=2\cos.\theta^2-1,$$

qui valores in aequatione priore substituti praebent aequationem identicam, ita ut utrique aequationi satisfiat sumendo $\cos.k\omega=\cos.\theta$.

3. Pro ω igitur eiusmodi angulum assumi oportet, ut fiat $\cos.k\omega=\cos.\theta$, unde quidem statim deducitur $k\omega=\theta$ ideoque $\omega=\frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem cosinum habentes, qui praeter ipsum angulum θ sunt $2\pi\pm\theta$, $4\pi\pm\theta$, $6\pi\pm\theta$ etc. atque adeo in genere $2i\pi\pm\theta$ denotante i omnes numeros integros, quaesito nostro satisfiet faciendo $k\omega=2i\pi\pm\theta$, unde colligitur angulus $\omega=\frac{2i\pi\pm\theta}{k}$, sicque pro ω nancisceremur innumerabiles angulos satisfacientes, quorum autem sufficiet tot assumsisse, quot exponens k continet unitates; successive igitur angulo ω sequentes tribuamus valores

$$\frac{\theta}{k},\quad \frac{2\pi+\theta}{k},\quad \frac{4\pi+\theta}{k},\quad \frac{6\pi+\theta}{k},\quad \frac{8\pi+\theta}{k},\quad \ldots\quad \frac{2(k-1)\pi+\theta}{k}.$$

Quodsi ergo angulo ω successive singulos istos valores, quorum numerus est $=k$, tribuamus, formula $1-2x\cos.\omega+xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1-2x^k\cos.\theta+x^{2k}$, quorum numerus erit $=k$.

4. Inventis iam omnibus factoribus duplicatis nostri denominatoris fractio $\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}$ resolvi debet in tot fractiones partiales, quarum deno-

minatores sint ipsi isti factores duplicati, quorum numerus est k, ita ut in genere talis fractio partialis habitura sit talem formam

$$\frac{A+Bx}{1-2x\cos.\omega+xx},$$

quam insuper resolvamus in binas simplices etsi imaginarias, et cum sit

$$xx-2x\cos.\omega+1=(x-\cos.\omega+\sqrt{-1}\cdot\sin.\omega)(x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega}+\frac{g}{x-\cos.\omega+\sqrt{-1}\cdot\sin.\omega},$$

ita ut totum resolutionis negotium huc redeat, ut ambo numeratores f et g determinentur; iis enim inventis habebitur summa ambarum fractionum

$$=\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}\cdot(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

ubi igitur erit

$$B=f+g \quad \text{et} \quad A=(f-g)\sqrt{-1}\cdot\sin.\omega-(f+g)\cos.\omega.$$

5. Per methodum igitur fractiones quascunque in fractiones simplices resolvendi statuamus

$$\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}=\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega}+R,$$

ubi R complectatur omnes reliquas fractiones partiales. Hinc per

$$x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$$

multiplicando habebitur

$$\frac{x^m-x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{1-2x^k\cos.\theta+x^{2k}}=f+R(x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega);$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$, ut membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita ut hinc nihil concludi posse videatur.

6. Hic igitur utamur regula notissima et loco tam numeratoris quam denominatoris eorum differentialia scribamus, unde nostra aequatio accipiet sequentem formam

$$\frac{mx^{m-1}-(m-1)x^{m-2}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^{k-1}\cos.\theta+2kx^{2k-1}}$$
$$=\frac{mx^{m}-(m-1)x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)}{-2kx^{k}\cos.\theta+2kx^{2k}}=f,$$

posito scilicet $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$. Tum autem erit

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et

$$x^{m-1}(\cos.\omega+\sqrt{-1}\cdot\sin.\omega)=x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et pro denominatore

$$x^k=\cos.k\omega+\sqrt{-1}\cdot\sin.k\omega \quad \text{et} \quad x^{2k}=\cos.2k\omega+\sqrt{-1}\cdot\sin.2k\omega;$$

unde fit numerator

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et denominator

$$-2k\cos.\theta\cos.k\omega+2k\cos.2k\omega-2k\sqrt{-1}\cdot\cos.\theta\sin.k\omega+2k\sqrt{-1}\cdot\sin.2k\omega.$$

7. Pro denominatore reducendo recordemur iam supra inventum esse $\cos.k\omega=\cos.\theta$, unde fit $\sin.k\omega=\sin.\theta$, tum vero

$$\cos.2k\omega=\cos.2\theta=2\cos.\theta^2-1 \quad \text{et} \quad \sin.2k\omega=2\sin.\theta\cos.\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\cos.\theta^2-2k+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta=-2k\sin.\theta^2+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta$$
$$=-2k\sin.\theta(\sin.\theta-\sqrt{-1}\cdot\cos.\theta),$$

quamobrem hoc valore adhibito habebimus

$$f=\frac{\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega}{2k\sin.\theta(\sqrt{-1}\cdot\cos.\theta-\sin.\theta)}.$$

Simul vero hinc sine novo calculo deducemus valorem g, quippe qui ab f ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g = \frac{\cos. m\omega - \sqrt{-1}\cdot \sin. m\omega}{-2k \sin.\theta\,(\sin.\theta + \sqrt{-1}\cdot\cos.\theta)}.$$

8. Inventis autem his litteris f et g pro litteris A et B colligemus primo

$$f+g = \frac{\cos.\theta \sin. m\omega - \sin.\theta \cos. m\omega}{k \sin.\theta} = \frac{\sin.(m\omega-\theta)}{k\sin.\theta},$$

tum vero erit

$$f-g = -\frac{\sqrt{-1}\cdot\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Ex his igitur reperiemus

$$B = \frac{\sin.(m\omega-\theta)}{k\sin.\theta}$$

et

$$A = \frac{\sin.\omega\cos.(m\omega-\theta) - \cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta} = -\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta},$$

ubi ergo imaginaria sponte se mutuo destruxerunt.

9. Inventis his valoribus A et B investigari oportet integrale partiale

$$\int \frac{(A+Bx)\,dx}{1-2x\cos.\omega + xx},$$

ubi, cum denominatoris differentiale sit

$$2x\,dx - 2\,dx\cos.\omega = 2\,dx(x-\cos.\omega),$$

statuamus

$$A + Bx = B(x-\cos.\omega) + C$$

eritque $C = A + B\cos.\omega$; hinc igitur erit

$$C = \frac{\cos.\omega\sin.(m\omega-\theta) - \sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Quia vero $-\sin.((m\omega-\theta)-\omega) = -\sin.(m\omega-\theta)\cos.\omega + \cos.(m\omega-\theta)\sin.\omega$, erit

$$C = \frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Hac ergo forma adhibita formula integranda $\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ discerpatur in has duas partes

$$\frac{B(x-\cos.\omega)\,dx}{1-2x\cos.\omega+xx}+\frac{C dx}{1-2x\cos.\omega+xx}.$$

Hic igitur prioris partis integrale manifesto est

$$Bl\sqrt{(1-2x\cos.\omega+xx)},$$

alterius vero partis facile patet integrale per arcum circuli expressum iri, cuius tangens sit $\frac{x\sin.\omega}{1-x\cos.\omega}$. Ad hoc integrale inveniendum ponamus

$$\int\frac{C dx}{1-2x\cos.\omega+xx}=D\,\text{A tang.}\,\frac{x\sin.\omega}{1-x\cos.\omega}$$

et sumtis differentialibus, quia $d.\,\text{A tang.}\,t$ aequale est $\frac{dt}{1+tt}$, habebimus

$$\frac{C dx}{1-2x\cos.\omega+xx}=D\,\frac{dx\sin.\omega}{1-2x\cos.\omega+xx},$$

unde manifesto fit

$$D=\frac{C}{\sin.\omega}=\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

10. Substituamus igitur loco B et D valores modo inventos et ex singulis factoribus denominatoris $1-2x^k\cos.\theta+x^{2k}$, quorum forma est $1-2x\cos.\omega+xx$, oritur pars integralis constans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}\,l\sqrt{(1-2x\cos.\omega+xx)}+\frac{\cos.(m\omega-\theta)}{k\sin.\theta}\,\text{A tang.}\,\frac{x\sin.\omega}{1-x\cos.\omega},$$

quae evanescit sumto $x=0$. In hac igitur forma tantum opus est, ut loco ω successive scribamus valores supra indicatos, scilicet

$$\omega=\frac{\theta}{k},\quad\frac{2\pi+\theta}{k},\quad\frac{4\pi+\theta}{k},\quad\frac{6\pi+\theta}{k}\quad\text{etc.},$$

donec perveniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim summa omnium harum formarum praebebit totum integrale indefinitum formulae propositae.

11. Postquam igitur integrale indefinitum elicuimus, nihil aliud superest, nisi ut in eo faciamus $x=\infty$, quo facto pars logarithmica ob

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega$$

erit $Bl(x-\cos.\omega)$. Est vero

$$l(x-\cos.\omega)=lx-\frac{\cos.\omega}{x}=lx$$

ob $\frac{\cos.\omega}{x}=0$; quamobrem facto $x=\infty$ quaelibet pars logarithmica habebit hanc formam $\frac{\sin.(m\omega-\theta)}{k\sin.\theta}lx$. Deinde pro partibus a circulo pendentibus facto $x=\infty$ fit

$$\frac{x\sin.\omega}{1-x\cos.\omega}=-\operatorname{tang}.\omega=\operatorname{tang}.(\pi-\omega)$$

sicque arcus, cuius haec est tangens, erit $=\pi-\omega$ hincque pars circularis quaecunque fiet $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$.

12. Cum quilibet valor anguli ω in genere hanc habeat formam $\frac{2i\pi+\theta}{k}$, erit angulus

$$m\omega-\theta=\frac{2im\pi-\theta(k-m)}{k}\quad\text{et}\quad \pi-\omega=\frac{\pi(k-2i)-\theta}{k}.$$

Ponamus brevitatis gratia

$$\frac{\theta(k-m)}{k}=\zeta\quad\text{et}\quad\frac{m\pi}{k}=\alpha,$$

ut sit

$$m\omega-\theta=2i\alpha-\zeta,$$

ubi loco i scribi debent successive numeri 0, 1, 2, 3 etc. usque ad $k-1$. Hinc igitur, si omnes partes logarithmicas in unam summam colligamus, ea ita repraesentari poterit

$$\frac{lx}{k\sin.\theta}\left\{\begin{matrix}-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\sin.(6\alpha-\zeta)\\ +\sin.(8\alpha-\zeta)+\cdots+\sin.(2(k-1)\alpha-\zeta)\end{matrix}\right\};$$

ubi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

13. Ad hoc ostendendum ponamus

$$S = -\sin.\zeta + \sin.(2\alpha-\zeta) + \sin.(4\alpha-\zeta) + \cdots + \sin.(2(k-1)\alpha-\zeta);$$

multiplicemus utrinque per $2\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\varphi = \cos.(\alpha-\varphi) - \cos.(\alpha+\varphi),$$

huius reductionis ope obtinebimus sequentem expressionem

$$\begin{aligned} 2S\sin.\alpha &= \cos.(\alpha+\zeta) \\ &- \cos.(\alpha-\zeta) - \cos.(3\alpha-\zeta) - \cos.(5\alpha-\zeta) - \cdots \\ &+ \cos.(\alpha-\zeta) + \cos.(3\alpha-\zeta) + \cos.(5\alpha-\zeta) + \cdots \\ &- \cos.((2k-1)\alpha-\zeta), \end{aligned}$$

unde deletis terminis se mutuo destruentibus habebitur

$$2S\sin.\alpha = \cos.(\alpha+\zeta) - \cos.((2k-1)\alpha-\zeta).$$

14. Ponamus hos duos angulos, qui sunt relicti,

$$\alpha+\zeta=p \quad \text{et} \quad (2k-1)\alpha-\zeta=q$$

eritque eorum summa $p+q=2\alpha k$. Quia porro est $\alpha=\frac{m\pi}{k}$, erit $p+q=2m\pi$, hoc est multiplo totius circuli peripheriae ob m numerum integrum. Quare cum sit $q=2m\pi-p$, erit $\cos.q=\cos.p$; unde patet summam inventam nihilo esse aequalem sicque manifestum est omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x=\infty$ se mutuo destruere.

15. Progrediamur igitur ad partes circulares, quarum forma generalis, ut vidimus, est $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$, quae posito $\alpha=\frac{m\pi}{k}$ et $\zeta=\frac{\theta(k-m)}{k}$ fit

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi+\theta}{k}\right) = \frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi}{k}-\frac{\theta}{k}\right).$$

Hic ponatur porro $\frac{\pi}{k}=\beta$ et $\pi-\frac{\theta}{k}=\gamma$, ut forma generalis sit

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma-2i\beta).$$

Quare si loco i scribamus ordine valores 0, 1, 2, 3, 4 usque ad $k-1$, omnes partes circulares hanc progressionem constituent

$$\frac{1}{k \sin.\theta}(\gamma \cos.\zeta + (\gamma - 2\beta)\cos.(2\alpha - \zeta) + (\gamma - 4\beta)\cos.(4\alpha - \zeta) + \cdots \\ + (\gamma - 2(k-1)\beta)\cos.(2(k-1)\alpha - \zeta)).$$

Ponamus igitur

$$S = \gamma \cos.\zeta + (\gamma - 2\beta)\cos.(2\alpha - \zeta) + (\gamma - 4\beta)\cos.(4\alpha - \zeta) + \cdots \\ + (\gamma - 2(k-1)\beta)\cos.(2(k-1)\alpha - \zeta),$$

ut summa omnium partium circularium sit $\frac{S}{k \sin.\theta}$, quae ergo praebebit valorem quaesitum formulae integralis propositae casu, quo post integrationem statuitur $x = \infty$, ita ut totum negotium in investigando valore ipsius S versetur.

16. Hunc in finem multiplicemus utrinque per $2\sin.\alpha$, et cum in genere sit

$$2\sin.\alpha \cos.\varphi = \sin.(\alpha + \varphi) - \sin.(\varphi - \alpha),$$

hac reductione in singulis terminis facta perveniemus ad hanc aequationem

$$\begin{aligned} 2S\sin.\alpha &= \gamma \sin.(\alpha + \zeta) \\ &+ \gamma \sin.(\alpha - \zeta) + (\gamma - 2\beta)\sin.(3\alpha - \zeta) + (\gamma - 4\beta)\sin.(5\alpha - \zeta) + \cdots \\ &- (\gamma - 2\beta)\sin.(\alpha - \zeta) - (\gamma - 4\beta)\sin.(3\alpha - \zeta) - (\gamma - 6\beta)\sin.(5\alpha - \zeta) - \cdots \\ &+ (\gamma - 2(k-1)\beta)\sin.((2k-1)\alpha - \zeta), \end{aligned}$$

ubi praeter primum et ultimum terminum omnes reliqui contrahi possunt, ita ut prodeat

$$2S\sin.\alpha = \gamma \sin.(\alpha + \zeta) + 2\beta \sin.(\alpha - \zeta) + 2\beta \sin.(3\alpha - \zeta) + 2\beta \sin.(5\alpha - \zeta) \\ + \cdots + 2\beta \sin.((2k-3)\alpha - \zeta) + (\gamma - 2(k-1)\beta)\sin.((2k-1)\alpha - \zeta).$$

17. Iam pro hac serie summanda ponamus porro

$$T = 2\sin.(\alpha - \zeta) + 2\sin.(3\alpha - \zeta) + 2\sin.(5\alpha - \zeta) + \cdots + 2\sin.((2k-3)\alpha - \zeta),$$

ut habeamus

$$2S\sin.\alpha = \gamma \sin.(\alpha + \zeta) + (\gamma - 2(k-1)\beta)\sin.((2k-1)\alpha - \zeta) + \beta T.$$

Iam multiplicemus ut hactenus per $\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\varphi = \cos.(\varphi-\alpha) - \cos.(\varphi+\alpha),$$

facta hac reductione nanciscimur

$$\begin{aligned} T\sin.\alpha = {} & +\cos.\zeta \\ & +\cos.(2\alpha-\zeta)+\cos.(4\alpha-\zeta)+\cdots+\cos.(2(k-2)\alpha-\zeta) \\ & -\cos.(2\alpha-\zeta)-\cos.(4\alpha-\zeta)-\cdots-\cos.(2(k-2)\alpha-\zeta) \\ & -\cos.(2(k-1)\alpha-\zeta), \end{aligned}$$

unde deletis terminis, quae se mutuo destruunt, remanebit tantum ista expressio

$$T\sin.\alpha = \cos.\zeta - \cos.(2(k-1)\alpha-\zeta).$$

Cum igitur sit $\alpha=\frac{m\pi}{k}$, erit $2(k-1)\alpha = 2m\pi - \frac{2m\pi}{k}$, cuius loco scribere licet $-\frac{2m\pi}{k}$, unde ob $\zeta=\frac{\theta(k-m)}{k}$ erit

$$T\sin.\alpha = \cos.\frac{\theta(k-m)}{k} - \cos.\frac{2m\pi+\theta(k-m)}{k}.$$

18. Nunc vero notetur in genere esse

$$\cos.p - \cos.q = 2\sin.\frac{q+p}{2}\sin.\frac{q-p}{2};$$

quare cum sit

$$p=\frac{\theta(k-m)}{k} \quad \text{et} \quad q=\frac{2m\pi+\theta(k-m)}{k},$$

erit

$$\frac{q+p}{2} = \frac{m\pi+\theta(k-m)}{k} \quad \text{et} \quad \frac{q-p}{2} = \frac{m\pi}{k},$$

unde sequitur fore

$$T\sin.\alpha = 2\sin.\frac{m\pi+\theta(k-m)}{k}\sin.\frac{m\pi}{k}$$

ideoque

$$T = 2\sin.\frac{m\pi+\theta(k-m)}{k}$$

ob $\alpha=\frac{m\pi}{k}$.

19. Hoc igitur valore T invento reperiemus porro

$$\begin{aligned} 2S\sin.\alpha = {} & \gamma\sin.(\alpha+\zeta) + (\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta) \\ & + 2\beta\sin.\frac{m\pi+\theta(k-m)}{k}, \end{aligned}$$

quae ob $\frac{m\pi+\theta(k-m)}{k}=\alpha+\zeta$ reducitur ad hanc formam

$$2S\sin.\alpha=(\gamma+2\beta)\sin.(\alpha+\zeta)+(\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta),$$

quae ita repraesentari potest

$$2S\sin.\alpha=(\gamma+2\beta)\big(\sin.(\alpha+\zeta)+\sin.((2k-1)\alpha-\zeta)\big)-2k\beta\sin.((2k-1)\alpha-\zeta),$$

ubi pro parte priore ob

$$\sin.p+\sin.q=2\sin.\frac{p+q}{2}\cos.\frac{p-q}{2}$$

erit

$$\frac{p+q}{2}=\alpha k \quad \text{et} \quad \frac{p-q}{2}=(k-1)\alpha-\zeta,$$

unde pars ipsa prior fit

$$2(\gamma+2\beta)\sin.\alpha k\cos.((k-1)\alpha-\zeta);$$

ubi cum sit $\alpha k=m\pi$, erit $\sin.\alpha k=0$, ita ut tantum supersit

$$2S\sin.\alpha=-2\beta k\sin.((2k-1)\alpha-\zeta)$$

hincque

$$S=-\frac{\beta k\sin.((2k-1)\alpha-\zeta)}{\sin.\alpha}.$$

Est vero

$$(2k-1)\alpha-\zeta=2m\pi-\frac{m\pi}{k}-\frac{\theta(k-m)}{k};$$

omisso termino $2m\pi$ erit igitur

$$S=+\frac{\pi\sin.\frac{m\pi+\theta(k-m)}{k}}{\sin.\frac{m\pi}{k}}$$

ideoque valor quaesitus erit

$$\frac{S}{k\sin.\theta}=+\frac{\pi\sin.\frac{m\pi+\theta(k-m)}{k}}{k\sin.\theta\sin.\frac{m\pi}{k}},$$

quae forma reducitur ad hanc

$$\frac{\pi\sin.\frac{m(\pi-\theta)+k\theta}{k}}{k\sin.\theta\sin.\frac{m\pi}{k}}.$$

20. Contemplemur hic ante omnia casum, quo $\theta=\frac{\pi}{2}$, et formula integralis proposita abit in hanc

$$\int\frac{x^{m-1}dx}{1+x^{2k}},$$

cuius ergo valor, si post integrationem ponatur $x=\infty$, evadet

$$=\frac{\pi\sin.\left(\frac{\pi}{2}+\frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}}=\frac{\pi\cos.\frac{m\pi}{2k}}{k\sin.\frac{m\pi}{k}}.$$

Quia igitur est $\sin.\frac{m\pi}{k}=2\sin.\frac{m\pi}{2k}\cos.\frac{m\pi}{2k}$, prodibit iste valor

$$=\frac{\pi}{2k\sin.\frac{m\pi}{2k}},$$

qui valor egregie convenit cum eo, quem non ita pridem pro formula $\int\frac{x^{m-1}dx}{1+x^k}$ assignavimus[1]), siquidem loco k scribatur $2k$.

21. Evolvamus etiam casum, quo $\theta=\pi$, et formula nostra integralis [abit in hanc]

$$\int\frac{x^{m-1}dx}{(1+x^k)^2},$$

cuius ergo facto $x=\infty$ valor erit

$$\frac{\pi\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{k\sin.\theta\sin.\frac{m\pi}{k}}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\sin.\theta}.$$

Huius autem posterioris fractionis casu $\theta=\pi$ tam numerator quam denominator evanescit; quare ut eius verus valor eruatur, loco utriusque eius differentiale scribamus, quo facto ista fractio abibit in hanc

$$\frac{d\theta\left(1-\frac{m}{k}\right)\cos.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{d\theta\cos.\theta},$$

cuius valor facto $\theta=\pi$ nunc manifesto est $1-\frac{m}{k}$; sicque valor integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti in superiore dissertatione[2]) invenimus.

1) Vide § 11 Commentationis praecedentis. A. G.
2) Vide § 17 Commentationis praecedentis. A. G.

22. Quo autem valorem generalem inventum commodiorem reddamus, ponamus $\pi - \theta = \eta$ fietque $\sin.\theta = \sin.\eta$ et $\cos.\theta = -\cos.\eta$; tum vero erit angulus

$$\frac{m(\pi-\theta)}{k} + \theta = \frac{m\eta}{k} + \pi - \eta,$$

cuius sinus est $\sin.\left(1-\frac{m}{k}\right)\eta$, unde valor quaesitus nostrae formulae erit

$$\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}},$$

atque hinc tandem sequens adepti sumus

THEOREMA

23. *Si haec formula integralis*

$$\int \frac{x^{m-1}dx}{1+2x^k \cos.\eta + x^{2k}}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extendatur, eius valor erit*

$$= \frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}},$$

sive cum sit

$$\sin.\left(1-\frac{m}{k}\right)\eta = \sin.\eta \cos.\frac{m\eta}{k} - \cos.\eta \sin.\frac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest

$$\frac{\pi \cos.\frac{m\eta}{k}}{k \sin.\frac{m\pi}{k}} - \frac{\pi \sin.\frac{m\eta}{k}}{k \operatorname{tang}.\eta \sin.\frac{m\pi}{k}}.$$

24. Consideremus nunc alio modo hanc formulam integralem

$$\int \frac{x^{m-1}dx}{1+2x^k \cos.\eta + x^{2k}},$$

cuius valor a termino $x=0$ usque ad $x=1$ ponatur $=P$, eiusdem vero

valor ab $x=1$ usque ad $x=\infty$ ponatur $=Q$, ita ut $P+Q$ exhibere debeat ipsum valorem ante inventum. Nunc vero pro valore Q inveniendo ponamus $x=\frac{1}{y}$ et formula nostra ita repraesentata

$$\frac{x^m}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{dx}{x}$$

ob $\frac{dx}{x}=-\frac{dy}{y}$ induet hanc formam

$$-\int\frac{y^{-m}}{1+2y^{-k}\cos.\eta+y^{-2k}}\cdot\frac{dy}{y}=-\int\frac{y^{2k-m-1}dy}{y^{2k}+2y^k\cos.\eta+1},$$

cuius valor a termino $y=1$ usque ad $y=0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q=+\int\frac{y^{2k-m-1}dy}{y^{2k}+2y^k\cos.\eta+1}$$

a termino $y=0$ usque ad $y=1$.

25. Quia in utraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 usque ad 1, nihil impedit, quominus in posteriore loco y scribamus x, unde pro $P+Q$ habebimus hanc formam integralem

$$\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos.\eta+x^{2k}}dx,$$

cuius valor a termino $x=0$ usque ad $x=1$ extensus aequabitur huic expressioni $\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\eta\sin.\frac{m\pi}{k}}$. Comparatis igitur his binis formulis integralibus nanciscemur sequens theorema notatu maxime dignum.

THEOREMA

26. *Haec formula integralis*

$$\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos.\eta+x^{2k}}dx$$

a termino $x=0$ usque ad terminum $x=1$ extensa aequalis est huic formulae integrali

$$\int\frac{x^{m-1}dx}{1+2x^k\cos.\eta+x^{2k}}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extensae; utriusque enim valor erit*

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\eta\sin.\frac{m\pi}{k}}.$$

27. Quodsi hanc fractionem $\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}$ in seriem infinitam evolvamus, quae sit

$$\sin.\eta+Ax^k+Bx^{2k}+Cx^{3k}+Dx^{4k}+Ex^{5k}+\text{etc.},$$

per denominatorem multiplicando perveniemus ad hanc expressionem infinitam

$$\begin{array}{lllllll}\sin.\eta-\sin.\eta+ & Ax^k+ & Bx^{2k}+ & Cx^{3k}+ & Dx^{4k}+ & Ex^{5k}+\text{etc.},\\ +2\sin.\eta\cos.\eta & +2A\cos.\eta & +2B\cos.\eta & +2C\cos.\eta & +2D\cos.\eta & +\text{etc.}\\ & +\sin.\eta & +A & +B & +C & +\text{etc.}\end{array}$$

unde singulis terminis ad nihilum reductis reperiemus

$$\begin{array}{lll}1.\ A+2\sin.\eta\cos.\eta=0 & \text{hincque} & A=-\sin.2\eta,\\ 2.\ B+2A\cos.\eta+\sin.\eta=0, & \text{unde fit} & B=\sin.3\eta,\\ 3.\ C+2B\cos.\eta+A=0, & \text{unde fit} & C=-\sin.4\eta,\\ 4.\ D+2C\cos.\eta+B=0, & \text{unde fit} & D=\sin.5\eta\\ \quad\text{etc.} & & \quad\text{etc.,}\end{array}$$

ita ut nostra fractio $\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}$ resolvatur in hanc seriem

$$\sin.\eta-x^k\sin.2\eta+x^{2k}\sin.3\eta-x^{3k}\sin.4\eta+x^{4k}\sin.5\eta-\text{etc.}$$

28. Multiplicemus nunc hanc seriem per

$$x^{m-1}dx+x^{2k-m-1}dx$$

et post integrationem faciamus $x=1$, ut obtineamus valorem huius formulae

$$\sin.\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos.\eta+x^{2k}}dx$$

pro casu $x=1$, hocque modo perveniemus ad geminas sequentes series

$$\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$

$$\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.}$$

Aggregatum igitur harum duarum serierum iunctim sumtarum aequabitur huic valori

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}},$$

unde subiungamus adhuc istud theorema.

THEOREMA

29. *Si η denotet angulum quemcunque, litterae vero m et k pro lubitu accipiantur ex iisque binae sequentes series formentur*

$$P=\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$

$$Q=\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.},$$

neutrius quidem summa exhiberi potest, utriusque autem iunctim sumtae summa erit

$$P+Q=\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}}.$$

COROLLARIUM

30. Quodsi ergo angulum η infinite parvum capiamus, ut fiat

$$\sin.\eta=\eta,\quad \sin.2\eta=2\eta,\quad \sin.3\eta=3\eta\quad\text{etc.},$$

quia in formula summae fiet

$$\sin.\left(1-\frac{m}{k}\right)\eta=\left(1-\frac{m}{k}\right)\eta,$$

si utrinque per η dividamus, obtinebimus sequentem seriem geminatam

$$\begin{aligned}&\frac{1}{m}-\frac{2}{m+k}+\frac{3}{m+2k}-\frac{4}{m+3k}+\frac{5}{m+4k}-\text{etc.}\\ +&\frac{1}{2k-m}-\frac{2}{3k-m}+\frac{3}{4k-m}-\frac{4}{5k-m}+\frac{5}{6k-m}-\text{etc.},\end{aligned}$$

cuius ergo summa erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k \sin.\frac{m\pi}{k}}$; ubi notetur ambas istas series non incongrue in hanc simplicem contrahi posse

$$\frac{2k}{m(2k-m)}-\frac{8k}{(m+k)(3k-m)}+\frac{18k}{(m+2k)(4k-m)}-\frac{32k}{(m+3k)(5k-m)}+\text{etc.},$$

ubi numeratores sunt numeri quadrati duplicati.

31. Formulae autem, quarum valores hactenus invenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis m scribamus $k-n$; tum enim in valore integrali invento fiet $\left(1-\frac{m}{k}\right)\eta=\frac{n\eta}{k}$, at vero pro denominatore fiet $\frac{m\pi}{k}=\pi-\frac{n\pi}{k}$, cuius sinus erit $\sin.\frac{n\pi}{k}$; sicque nostra formula inventa hanc induet formam $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$, quae ergo exprimet valorem huius formulae integralis

$$\int \frac{x^{k-n-1}dx}{1+2x^k \cos.\eta + x^{2k}}$$

ab $x=0$ usque ad $x=\infty$, ut et huius formulae

$$\int \frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k \cos.\eta + x^{2k}}dx$$

a termino $x=0$ usque ad terminum $x=1$; et quia utriusque valor est $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$, perspicuum est eum manere eundem, etsi loco n scribatur $-n$, ex quo prior formula ita repraesentari poterit

$$\int \frac{x^{k\pm n-1}dx}{1+2x^k \cos.\eta + x^{2k}};$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

32. Ponendo $m=k-n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\sin.\eta}{k-n}-\frac{\sin.2\eta}{2k-n}+\frac{\sin.3\eta}{3k-n}-\frac{\sin.4\eta}{4k-n}+\text{etc.}$$
$$+\frac{\sin.\eta}{k+n}-\frac{\sin.2\eta}{2k+n}+\frac{\sin.3\eta}{3k+n}-\frac{\sin.4\eta}{4k+n}+\text{etc.},$$

cuius ergo summa erit $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}}$. Tum vero si hae geminae series in unam contrahantur et utrinque per $2k$ dividatur, obtinebitur sequens summatio memoratu digna

$$\frac{\pi\sin.\frac{n\eta}{k}}{2kk\sin.\frac{n\pi}{k}}=\frac{\sin.\eta}{kk-nn}-\frac{2\sin.2\eta}{4kk-nn}+\frac{3\sin.3\eta}{9kk-nn}-\frac{4\sin.4\eta}{16kk-nn}+\text{etc.}$$

33. Quodsi haec postrema series differentietur sumendo solum angulum η variabilem, ob $d\sin.\frac{n\eta}{k}=\frac{nd\eta}{k}\cos.\frac{n\eta}{k}$ habebimus

$$\frac{\pi n\cos.\frac{n\eta}{k}}{2k^3\sin.\frac{n\pi}{k}}=\frac{\cos.\eta}{kk-nn}-\frac{4\cos.2\eta}{4kk-nn}+\frac{9\cos.3\eta}{9kk-nn}-\frac{16\cos.4\eta}{16kk-nn}+\text{etc.}$$

Unde si sumatur $\eta=0$, orietur ista summatio

$$\frac{\pi n}{2k^3\sin.\frac{n\pi}{k}}=\frac{1}{kk-nn}-\frac{4}{4kk-nn}+\frac{9}{9kk-nn}-\frac{16}{16kk-nn}+\text{etc.};$$

sin autem sumatur $\eta=90^0=\frac{\pi}{2}$, erit

$$\cos.\eta=0,\quad \cos.2\eta=-1,\quad \cos.3\eta=0,\quad \cos.4\eta=+1\ \text{etc.},$$

unde nascitur sequens series

$$\frac{n\pi\cos.\frac{n\pi}{2k}}{2k^3\sin.\frac{n\pi}{k}}=\frac{4}{4kk-nn}-\frac{16}{16kk-nn}+\frac{36}{36kk-nn}-\frac{64}{64kk-nn}+\text{etc.}$$

Quia autem $\sin.\frac{n\pi}{k}=2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, erit eiusdem seriei summa $\frac{n\pi}{4k^3\sin.\frac{n\pi}{2k}}$.

34. At si series illa § 32 exhibita in $d\eta$ ducatur et integretur, ob $\int d\eta \sin.\frac{n\eta}{k} = -\frac{k}{n}\cos.\frac{n\eta}{k}$ erit

$$C - \frac{\pi \cos.\frac{n\eta}{k}}{2nk \sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} - \text{etc.}$$

Ut autem hic constantem addendam C definiamus, sumamus $\eta = 0$ fietque

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.};$$

quare si huius seriei summa aliunde pateat, constans C definiri poterit. Series autem haec in sequentem geminatam resolvi potest

$$2nC - \frac{\pi}{k \sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$
$$- \frac{1}{k-n} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.}$$

35. Cum igitur in *Introductione in Analysin Infinitorum*[1]) pag. 142 ad hanc pervenissem seriem

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.} = \frac{\pi}{2kn \sin.\frac{n\pi}{k}} - \frac{1}{2nn}$$

(hic scilicet loco litterarum ibi adhibitarum m et n scripsi n et k), hoc valore adhibito nostra aequatio erit

$$C - \frac{\pi}{2nk \sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk \sin.\frac{n\pi}{k}},$$

unde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem

$$\frac{\pi \cos.\frac{n\eta}{k}}{2nk \sin.\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos.\eta}{kk-nn} - \frac{\cos.2\eta}{4kk-nn} + \frac{\cos.3\eta}{9kk-nn} - \frac{\cos.4\eta}{16kk-nn} + \text{etc.},$$

quae series utique omni attentione digna videtur.

1) Vide *Introductionem in analysin infinitorum*, t. I cap. X; LEONHARDI EULERI *Opera omnia*, series I, vol. 8. A. G.

METHODUS INVENIENDI FORMULAS INTEGRALES QUAE CERTIS CASIBUS DATAM INTER SE TENEANT RATIONEM UBI SIMUL METHODUS TRADITUR FRACTIONES CONTINUAS SUMMANDI

Commentatio 594 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 178—216

1. Quemadmodum in seriebus recurrentibus quilibet terminus ex uno pluribusve praecedentibus secundum legem quandam constantem determinatur, ita hic eiusmodi series sum consideraturus, in quibus quilibet terminus ex uno pluribusve praecedentibus secundum quampiam legem variabilem determinatur. Quoniam autem in talibus seriebus formula generalis singulos terminos exprimens plerumque non est algebraica sed transcendens, singulos terminos per formulas integrales exhiberi conveniet; quae ut valores determinatos praebeant, post integrationem quantitati variabili valorem determinatum tribui assumo, ita ut singuli termini prodeant quantitates determinatae; atque nunc quaestio principalis huc redit, quemadmodum istae formulae integrales debeant esse comparatae, ut quilibet terminus secundum datam legem ex uno pluribusve praecedentibus determinetur.

2. Quod quo clarius perspiciatur, contemplemur seriem notissimam harum formularum integralium

$$\int \frac{dx}{\sqrt{(1-xx)}}, \int \frac{xx\,dx}{\sqrt{(1-xx)}}, \int \frac{x^4\,dx}{\sqrt{(1-xx)}}, \int \frac{x^6\,dx}{\sqrt{(1-xx)}} \quad \text{etc.};$$

quae si singulae ita integrentur, ut evanescant posito $x = 0$, tum vero variabili x tribuatur valor $= 1$, quilibet terminus a praecedente ita pendet, ut sit

$$\int \frac{xx\,dx}{\sqrt{(1-xx)}} = \frac{1}{2}\int \frac{dx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^4dx}{\sqrt{(1-xx)}} = \frac{3}{4}\int \frac{xx\,dx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^6dx}{\sqrt{(1-xx)}} = \frac{5}{6}\int \frac{x^4dx}{\sqrt{(1-xx)}}$$

atque in genere

$$\int \frac{x^n dx}{\sqrt{(1-xx)}} = \frac{n-1}{n}\int \frac{x^{n-2}dx}{\sqrt{(1-xx)}}.$$

Unde patet hanc formulam generalem spectari posse tanquam terminum generalem illius seriei atque quemlibet terminum ex praecedente oriri, si iste multiplicetur per $\frac{n-1}{n}$.

3. Ad similitudinem igitur huius casus seriem formularum integralium ita in genere constituamus

$$\int dv, \quad \int x\,dv, \quad \int xx\,dv, \quad \int x^3dv, \quad \int x^4dv \quad \text{etc.},$$

ita ut terminus indici n respondens sit

$$\int x^{n-1}dv,$$

quae singula integralia ita accipi sumamus, ut evanescant posito $x = 0$; post integrationem autem quantitati variabili x tribuamus quempiam valorem constantem, veluti $x = 1$ vel alii cuipiam numero. Quibus positis quaestio huc redit, qualis pro v assumi debeat functio ipsius x, ut quilibet terminus per unum vel duos pluresve praecedentes secundum legem quandam datam, utcunque variabilem sive ab indice n pendentem, determinetur; ubi quidem imprimis eo erit respiciendum, ad quot dimensiones index n in scala relationis proposita ascendat; plerumque autem non ultra primam dimensionem assurgere erit opus. Hinc igitur sequentia problemata pertractemus.

PROBLEMA 1

4. *Invenire functionem v, ut ista relatio inter binos terminos sibi succedentes locum habeat*

$$\int x^n dv = \frac{\alpha n + a}{\beta n + b}\int x^{n-1} dv.$$

SOLUTIO

Requiritur igitur hic, ut sit

$$(\alpha n + a)\int x^{n-1} dv = (\beta n + b)\int x^n dv,$$

si scilicet post integrationem variabili x certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili x iste valor constans fuerit datus, ponamus in genere, dum x est variabilis, hanc aequationem locum habere

$$(\alpha n + a)\int x^{n-1} dv = (\beta n + b)\int x^n dv + V,$$

quantitatem autem V ita esse comparatam, ut evanescat, postquam variabili ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, ut evanescant posito $x = 0$, necesse est, ut etiam ista quantitas V eodem quoque casu evanescat.

5. Quoniam haec aequalitas subsistere debet pro omnibus indicibus n, quos quidem semper ut positivos spectamus, facile intelligitur quantitatem istam V factorem habere debere x^n; quo pacto iam isti conditioni satisfit, ut posito $x = 0$ etiam fiat $V = 0$. Quamobrem statuamus

$$V = x^n Q,$$

ubi Q denotet functionem ipsius x proposito accommodatam, et quam simul ita comparatam esse desideramus, ut evanescat, si ipsi x certus quidam valor tribuatur.

6. Cum igitur esse debeat

$$(\alpha n + a)\int x^{n-1} dv = (\beta n + b)\int x^n dv + x^n Q,$$

differentietur ista aequatio ac differentiali per x^{n-1} diviso pervenietur ad hanc aequationem differentialem

$$(\alpha n + a)dv = (\beta n + b)xdv + nQdx + xdQ;$$

quae cum subsistere debeat pro omnibus valoribus ipsius n, termini ista littera affecti seorsim se tollere debent, unde nanciscimur has duas aequalitates

$$\text{I. } (\alpha - \beta x)dv = Qdx \quad \text{et} \quad \text{II. } (a - bx)dv = xdQ.$$

Ex priore fit $dv = \frac{Qdx}{\alpha - \beta x}$, ex altera vero $dv = \frac{xdQ}{a - bx}$, qui duo valores inter se aequati suppeditant hanc aequationem $\frac{dQ}{Q} = \frac{dx}{x} \cdot \frac{a - bx}{\alpha - \beta x}$, quae aequatio resolvitur in has partes

$$\frac{dQ}{Q} = \frac{a}{\alpha} \cdot \frac{dx}{x} + \frac{a\beta - b\alpha}{\alpha} \cdot \frac{dx}{\alpha - \beta x},$$

cuius ergo integrale erit

$$lQ = \frac{a}{\alpha} lx - \frac{a\beta - b\alpha}{\alpha\beta} l(\alpha - \beta x),$$

unde deducitur

$$Q = Cx^{\frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

7. Ex hoc valore pro Q invento statim patet eum evanescere casu $x = \frac{\alpha}{\beta}$, si modo fuerit $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; sin autem secus eveniat, non patet, quomodo haec quantitas ullo casu evanescere queat. Invento autem hoc valore Q inde reperietur

$$dv = Cx^{\frac{a}{\alpha}} dx(\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

hincque nostrae seriei terminus indici n respondens erit

$$\int x^{n-1} dv = C\int x^{n + \frac{a}{\alpha} - 1} dx(\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1},$$

tum vero erit

$$V = Cx^{n + \frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

Ubi res imprimis eo redit, ut ista quantitas praeter casum $x = 0$ insuper alio casu evanescat.

COROLLARIUM 1

8. Hic duo casus occurrunt, qui peculiarem evolutionem postulant; prior est, quo $\alpha = 0$; tum autem inchoandum erit ab aequatione $\frac{dQ}{Q} = -\frac{(a - bx)dx}{\beta xx}$, unde integrando elicitur $lQ = \frac{a}{\beta x} + \frac{b}{\beta} lx$, hincque sumendo e pro numero, cuius logarithmus hyperbolicus $= 1$, colligitur

$$Q = e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}},$$

quae formula in nihilum abire nequit, nisi fiat $\frac{a}{\beta x} = -\infty$ ideoque $x = 0$, sicque non duo haberentur casus, quibus fieret $V = 0$, cum tamen duo desiderentur. Interim autem hinc fiet

$$dv = \frac{e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}} dx}{-\beta x}.$$

COROLLARIUM 2

9. Alter casus peculiarem integrationem postulans erit $\beta = 0$; tum autem erit $\frac{dQ}{Q} = \frac{dx(a - bx)}{\alpha x}$, unde fit $lQ = \frac{a}{\alpha} lx - \frac{bx}{\alpha}$ ideoque $Q = x^{\frac{a}{\alpha}} e^{\frac{-bx}{\alpha}}$, quae formula casu $x = \infty$ evanescit, si modo fuerit $\frac{b}{\alpha}$ numerus positivus; sin autem $\frac{b}{\alpha}$ fuerit numerus negativus, tum Q evanescit casu $x = -\infty$. Porro vero hoc casu fiet

$$dv = \frac{x^{\frac{a}{\alpha}} e^{\frac{-bx}{\alpha}} dx}{\alpha}.$$

SCHOLION

9[a][1]). His in genere observatis aliquot casus speciales evolvamus, quibus litteris α, β et a, b certos valores tribuemus, qui ad casus iam satis cognitos perducant.

EXEMPLUM 1

9[b][1]). *Quaerantur formulae integrales, ut fiat*

$$\int x^n dv = \frac{2n-1}{2n} \int x^{n-1} dv.$$

1) In editione principe falso numeri 8 et 9 iterantur. A. G.

Cum igitur hic esse debeat $(2n-1)\int x^{n-1}dv = 2n\int x^n dv$, erit hoc casu $\alpha = 2$ et $a = -1$, tum vero $\beta = 2$ et $b = 0$; hinc fit

$$\frac{dQ}{Q} = -\frac{dx}{2x(1-x)} = -\frac{dx}{2x} - \frac{dx}{2(1-x)},$$

inde integrando

$$lQ = -\frac{1}{2}lx + \frac{1}{2}l(1-x)$$

ideoque

$$Q = C\sqrt{\frac{1-x}{x}}, \quad \text{ergo} \quad V = Cx^n\sqrt{\frac{1-x}{x}}.^{1)}$$

Porro cum hic sit $dv = \frac{Qdx}{2(1-x)}$, erit

$$dv = \frac{Cdx\sqrt{\frac{1-x}{x}}}{2(1-x)} = \frac{Cdx}{2\sqrt{(x-xx)}};$$

sumto ergo $C = 2$ erit $dv = \frac{dx}{\sqrt{(x-xx)}}$ et formula nostra generalis

$$\int x^{n-1}dv = \int\frac{x^{n-1}dx}{\sqrt{(x-xx)}};$$

unde cum sit $V = x^n\sqrt{\frac{1-x}{x}}$, haec quantitas manifesto evanescit sumto $x = 1$, ita ut nostra formula, si post integrationem statuatur $x = 1$, quaesito satisfaciat. Quodsi iam ponamus $x = yy$, ista formula induet hanc formam

$$2\int\frac{y^{2n-2}dy}{\sqrt{(1-yy)}},$$

quae posito post integrationem $y = 1$ praebet hanc relationem

$$\int\frac{y^{2n}dy}{\sqrt{(1-yy)}} = \frac{2n-1}{2n}\int\frac{y^{2n-2}dy}{\sqrt{(1-yy)}},$$

1) In editione principe factor C huic formulae deest. Factores autem constantes EULERUS etiam in formulis sequentibus saepenumero neglexit. A. G.

quae continet relationes supra (§ 2) commemoratas; hinc enim fiet

$$\int\frac{yydy}{\sqrt{(1-yy)}}=\frac{1}{2}\int\frac{dy}{\sqrt{(1-yy)}},$$

$$\int\frac{y^4dy}{\sqrt{(1-yy)}}=\frac{3}{4}\int\frac{yydy}{\sqrt{(1-yy)}}$$

et

$$\int\frac{y^6dy}{\sqrt{(1-yy)}}=\frac{5}{6}\int\frac{y^4dy}{\sqrt{(1-yy)}}.$$

EXEMPLUM 2

10. *Quaerantur formulae integrales, ut fiat*

$$\int x^n dv=\frac{\alpha n-1}{\alpha n}\int x^{n-1}dv.$$

Cum igitur hic esse debeat $(\alpha n-1)\int x^{n-1}dv=\alpha n\int x^n dv$, erit hoc casu $a=-1$, $\beta=\alpha$ et $b=0$, unde per formulas supra datas colligitur

$$Q=Cx^{\frac{-1}{\alpha}}(\alpha-\alpha x)^{\frac{-\alpha}{\alpha}}=Cx^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}},$$

quae quantitas manifesto evanescit posito $x=1$. Tum autem erit

$$dv=\frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}dx}{1-x},$$

unde formula nostra generalis erit

$$\int x^{n-1}dv=\int x^{n-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}dx=\int\frac{x^{n-\frac{1}{\alpha}-1}dx}{(1-x)^{1-\frac{1}{\alpha}}},$$

quae concinnior redditur faciendo $x=y^\alpha$; tum enim ea induet hanc formam

$$\int\frac{y^{\alpha n-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}},$$

ubi iterum post integrationem statui debet $y=1$. Erit hinc

$$\int\frac{y^{\alpha n+\alpha-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}=\frac{\alpha n-1}{\alpha n}\int\frac{y^{\alpha n-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}.$$

atque hinc orientur sequentes casus speciales

$$\int \frac{y^{2\alpha-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha}\int \frac{y^{\alpha-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}$$

et

$$\int \frac{y^{3\alpha-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha}\int \frac{y^{2\alpha-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}.$$

11. Hinc igitur si sumatur $\alpha = 1$, ut fieri debeat

$$\int x^n dv = \frac{n-1}{n}\int x^{n-1}dv,$$

formula nostra generalis iam in y expressa erit $\int y^{n-2}dy$, cuius ergo valor est $\frac{1}{n-1}y^{n-1} = \frac{1}{n-1}$, unde tota series nostrarum formularum integralium abibit in hanc

$$\frac{1}{0},\quad \frac{1}{1},\quad \frac{1}{2},\quad \frac{1}{3},\quad \frac{1}{4},\quad \frac{1}{5},\quad \frac{1}{6},\quad \frac{1}{7}\quad \text{etc.}$$

12. Sumamus etiam $\alpha = \frac{1}{2}$ et iam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$Q = \frac{(1-x)^2}{xx} \quad \text{et} \quad dv = \frac{(1-x)dx}{xx},$$

unde formula nostra generalis fit

$$\int x^{n-1}dv = \int x^{n-3}(1-x)\,dx,$$

cuius ergo valor algebraice expressus erit

$$\frac{1}{n-2}x^{n-2} - \frac{1}{n-1}x^{n-1} = \frac{1}{(n-1)(n-2)},$$

unde series nostrarum formularum evadet

$$\frac{1}{0\cdot-1},\quad \frac{1}{0\cdot 1},\quad \frac{1}{1\cdot 2},\quad \frac{1}{2\cdot 3},\quad \frac{1}{3\cdot 4},\quad \frac{1}{4\cdot 5}\quad \text{etc.}$$

EXEMPLUM 3

13. *Quaerantur formulae integrales, ut sit*

$$\int x^n\,dv = n\int x^{n-1}\,dv.$$

Cum igitur esse debeat $n\int x^{n-1}dv = 1\int x^n dv$, erit $\alpha = 1$, $a = 0$, $b = 1$, $\beta = 0$. Cum igitur sit $\beta = 0$, casus Corollarii 2 hic locum habet indeque erit $Q = e^{-x}$ ideoque $V = e^{-x}x^n$, quae quantitas his duobus casibus evanescit $x = 0$ et $x = \infty$. Porro vero erit $dv = e^{-x}dx$ hincque formula nostra generalis fiet $\int x^{n-1}dx\,e^{-x}$, unde ipsi seriei termini ab initio sequenti modo se habebunt:

$$\int e^{-x}dx,\quad \int e^{-x}x\,dx,\quad \int e^{-x}xx\,dx,\quad \int e^{-x}x^3dx \text{ etc.},$$

quibus integratis ita, ut evanescant posito $x = 0$, tum vero posito $x = \infty$ orietur sequens series satis simplex

$$1,\quad 1,\quad 1\cdot 2,\quad 1\cdot 2\cdot 3,\quad 1\cdot 2\cdot 3\cdot 4,\quad 1\cdot 2\cdot 3\cdot 4\cdot 5 \quad \text{etc.},$$

quae est series hypergeometrica WALLISII[1]), cuius ergo terminus generalis est

$$\int x^{n-1}e^{-x}dx = 1\cdot 2\cdot 3\cdot 4\cdots(n-1).$$

14. Ope ergo huius termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{3}{2}$ ac valor huius termini erit $\int e^{-x}dx\sqrt{x}$, cuius autem valor nullo modo algebraice exprimi potest. Inveni autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$ denotante π peripheriam circuli, cuius diameter $= 1$, unde hic vicissim cognoscimus esse $\int e^{-x}dx\sqrt{x} = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens indici $\frac{1}{2}$ respondens erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int\frac{e^{-x}dx}{\sqrt{x}}$. Quodsi hic ponamus $e^x = y$, ita ut posito $x = 0$

1) Vide I. WALLISII *Arithmeticam infinitorum,* Oxonii 1656, scholium adiectum ad propositionem 190; *Opera,* t. I, Oxoniae 1695, p. 466. Quo in scholio *progressio hypergeometrica* vocatur „progressio facta ex termini primi continua multiplicatione in quotlibet succedentes numeros inaequales, sive crescentes sive decrescentes; (puta 1, 2, 6, 24, etc. ex continue multiplicatis $1\times 2\times 3\times 4$ etc. vel $1, \frac{3}{2}, \frac{15}{8}, \frac{105}{48}$, etc. ex continue multiplicatis $1\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}$ etc.)" A. G.

sit $y=1$, at posito $x=\infty$ fiat $y=\infty$, tum ergo ista formula $\int\frac{e^{-x}dx}{\sqrt{x}}$ abit in hanc $\int\frac{dy}{yy\sqrt{ly}}$, quae formula, si ita integretur, ut evanescat posito $y=1$, tum vero fiat $y=\infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y=\frac{1}{z}$, erunt termini integrationis $z=1$ et $z=0$ et formula integralis erit

$$-\int\frac{dz}{\sqrt{-lz}}\begin{bmatrix}\text{a} & z=1\\ \text{ad} & z=0\end{bmatrix}=\sqrt{\pi}$$

sive permutatis terminis integrationis erit

$$\int\frac{dz}{\sqrt{-lz}}\begin{bmatrix}\text{a} & z=0\\ \text{ad} & z=1\end{bmatrix}=\sqrt{\pi},$$

quemadmodum iam olim observavi.[1])

EXEMPLUM 4

15. *Quaerantur formulae integrales, ut sit*

$$\int x^n dv=\frac{1}{n}\int x^{n-1}dv \quad \textit{sive} \quad \int x^{n-1}dv=n\int x^n dv.$$

Hic est $\alpha=0$ et $a=1$, $\beta=1$ et $b=0$; qui ergo est casus in Corollario 1 tractatus, unde colligitur fore $Q=e^{\frac{1}{x}}$ ideoque $V=x^n e^{\frac{1}{x}}$, quae formula nequidem evanescit sumto $x=0$, quandoquidem formula $e^{\frac{1}{0}}$ aequivalet infinito infinitesimae potestatis. Hic autem miro modo evenit, ut casus $x=-0$ reddat formulam $e^{\frac{-1}{0}}$ subito evanescentem. Scilicet si ω denotet quantitatem infinite parvam, erit $e^{\frac{1}{\omega}}=\infty^\infty$, tum vero repente fiet $e^{\frac{-1}{\omega}}=\frac{1}{\infty^\infty}=0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $dv=-e^{\frac{1}{x}}\frac{dx}{x}$, ita ut formula nostra generalis futura sit $-\int x^{n-2}dxe^{\frac{1}{x}}$, quae autem nobis nullum usum praestare potest.

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1) Vide Commentationem 19 (indicis ENESTROEMIANI): *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt*, Comment. acad. sc. Petrop. **5** (1730/1), 1738, p. 36; *LEONHARDI EULERI Opera omnia*, series I, vol. 14; vide porro Commentationem 421: *Evolutio formulae integralis* $\int x^{f-1}dx\,(lx)^{\frac{m}{n}}$ *integratione a valore* $x=0$ *ad* $x=1$ *extensa*, Novi comment. acad. sc. Petrop. **16** (1771), 1772, p. 91; *LEONHARDI EULERI Opera omnia*, series I, vol. **17**. A. G.

16. Quodsi hic ponamus $\frac{1}{x}=y$, formula ista generalis transit in hanc $+\int\frac{e^y dy}{y^n}$. At vero nunc erit $V=\frac{e^y}{y^n}$, quae formula evanescit posito $y=-\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resolvere licebit. Sit enim seriei, quam quaerimus, primus terminus $=\omega$, ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4}, \;\cdots & \frac{\omega}{1\cdot 2\cdot 3\cdots(n-1)}. \end{array}$$

Supra autem vidimus huius formulae $1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(n-1)$ valorem exprimi per hoc integrale $\int x^{n-1}e^{-x}dx$ integratione ab $x=0$ ad $x=\infty$ extensa; tantum igitur opus est, ut hanc formulam integralem in denominatorem transferamus, et seriei, quam quaerimus, terminus generalis erit

$$\frac{1}{\int x^{n-1}e^{-x}dx},$$

unde satis intelligitur negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus evanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n+a}{\beta n+b}$ invertere atque formulam integralem in denominatorem transferre.

SCHOLION

17. Nisi sit vel $\alpha=0$ vel $\beta=0$, quos casus iam expedivimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales unitati. Cum enim esse debeat

$$\int x^n dv=\frac{\alpha n+a}{\beta n+b}\int x^{n-1}dv,$$

ponatur $x=\frac{\alpha y}{\beta}$ fietque

$$\frac{\alpha}{\beta}\int y^n dv=\frac{\alpha n+a}{\beta n+b}\int y^{n-1}dv,$$

quae aequatio reducitur ad hanc formam

$$\int y^n dv=\frac{n+a:\alpha}{n+b:\beta}\int y^{n-1}dv.$$

Quodsi iam nunc loco $\frac{a}{\alpha}$ scribamus a et b loco $\frac{b}{\beta}$, resolvenda erit haec formula

$$\int y^n dv = \frac{n+a}{n+b}\int y^{n-1} dv,$$

cuius resolutio, si loco x scribamus y et loco litterarum α et β unitatem, ex superiori solutione praebet primo

$$Q = Cy^a(1-y)^{b-a},$$

quod ergo evanescit posito $y = 1$, si modo fuerit $b > a$; tum autem erit ipsa formula

$$\int y^{n-1}dv = C\int y^{n+a-1}dy(1-y)^{b-a-1};$$

sin autem fuerit $b < a$, haec solutio, uti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma $\frac{1}{\int y^{n-1}dv}$, ita ut tum esse debeat

$$\frac{1}{\int y^n dv} = \frac{n+a}{n+b}\cdot\frac{1}{\int y^{n-1}dv}$$

sive

$$\int y^n dv = \frac{n+b}{n+a}\int y^{n-1} dv,$$

cuius resolutio permutatis litteris a et b praebet

$$Q = Cy^b(1-y)^{a-b},$$

quae iam casu $y = 1$ evanescit, si fuerit $a > b$; atque tum erit formula generalis

$$\int y^{n-1}dv = C\int y^{n+b-1}dy(1-y)^{a-b-1}.$$

Sive igitur sit $b > a$ sive $a > b$, solutio nulla amplius laborat difficultate.

18. Sin autem fuerit vel $\alpha = 0$ vel $\beta = 0$, loco alterius etiam scribi poterit unitas; unde si esse debeat

$$\int x^n dv = \frac{n+a}{b}\int x^{n-1}dv,$$

ob $\alpha = 1$ et $\beta = 0$ solutio nostra generalis dat

$$\frac{dQ}{Q} = \frac{dx}{x}(a-bx),$$

unde colligitur $Q = Cx^a e^{-bx}$, quae formula evanescit posito $x = \infty$, si modo b fuerit numerus positivus; tum autem fit terminus generalis

$$\int x^{n-1}\,dv = C\int x^{n+a-1}\,dx e^{-bx}.$$

At vero numerus b negativus esse nequit, quia alioquin conditio praescripta esset incongrua.

19. Consideremus etiam alterum casum, quo $\alpha = 0$ et $\beta = 1$ ideoque conditio praescripta

$$\int x^n dv = \frac{a}{n+b}\int x^{n-1}dv,$$

unde fit

$$\frac{dQ}{Q} = -\frac{dx}{xx}(a - bx).$$

Hinc autem pro Q orietur valor, qui praeter casum $x = 0$ evanescere non posset; quam ob causam formula generalis statui debet $\frac{1}{\int x^{n-1}dv}$, ita ut esse debeat

$$\int x^n dv = \frac{n+b}{a}\int x^{n-1}dv,$$

unde prodit

$$\frac{dQ}{Q} = \frac{dx}{x}(b - ax) \quad \text{ideoque} \quad Q = Ce^{-ax}x^b,$$

quae expressio evanescit posito $x = \infty$, quoniam a necessario debet esse numerus positivus; tum autem erit

$$dv = Ce^{-ax}x^b dx,$$

unde formula generalis seriei erit

$$\frac{1}{C\int x^{n+b-1}dx e^{-ax}}.$$

PROBLEMA 2

20. *Denotet T terminum indici n respondentem in serie, quam considerandam suscepimus, at vero T' terminum sequentem atque proponatur haec conditio adimplenda*

$$T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')}T.$$

SOLUTIO

Quoniam hic valores geminati occurrunt, huic conditioni commodissime satisfiet, si terminus generalis T tanquam productum ex duobus factoribus spectetur. Statuatur igitur $T = RS$ sitque terminus sequens $= R'S'$ et quaerantur formulae R et S, ut fiat

$$R' = \frac{\alpha n + a}{\beta n + b} R \quad \text{et} \quad S' = \frac{\alpha' n + a'}{\beta' n + b'} S;$$

tum enim sumendo $T = RS$ conditioni praescriptae manifesto satisfiet. Hoc igitur modo pro R et S vel huiusmodi formulae $\int x^{n-1} dv$ vel inversae $\frac{1}{\int x^{n-1} dv}$ reperientur, id quod pro solutione generali sufficit, unde rem exemplo illustremus.

EXEMPLUM

21. *Quaeratur formula generalis T, ut fiat*

$$T' = \frac{nn - cc}{nn} T.$$

Resolvamus igitur T in duos factores R et S ac statuamus

$$R' = \frac{n - c}{n} R \quad \text{et} \quad S' = \frac{n + c}{n} S.$$

Pro priore forma si statuamus $R = \int x^{n-1} dv$, ex solutione generali, ubi erit $\alpha = 1$, $a = -c$, $\beta = 1$ et $b = 0$, fiet

$$Q = Cx^{-c}(1 - x)^c,$$

quae forma manifesto evanescit posito $x = 1$; hincque quia fit

$$V = Cx^{n-c}(1 - x)^c,$$

haec forma etiam casu $x = 0$ evanescit, si modo n fuerit $> c$, id quod tuto assumi potest, quia exponentem n successive in infinitum crescere assumimus ac plerumque pro c fractiones tantum accipi solent. Hinc ergo erit

$$R = C\int x^{n-c-1}(1 - x)^{c-1} dx.$$

22. Hinc iam alter valor litterae S deduci posset scribendo tantum $-c$ loco c, tum autem non amplius fieret $Q=0$ posito $x=1$, quamobrem pro S formulam inversam $\frac{1}{\int x^{n-1}dv}$ assumi oportet, ut esse debeat

$$\int x^n dv = \frac{n}{n+c}\int x^{n-1}dv;$$

ubi cum sit $\alpha=1$, $a=0$, $\beta=1$ et $b=c$, reperitur

$$Q=C(1-x)^c,$$

quae forma manifesto fit $=0$ posito $x=1$; hinc autem prodit

$$dv=C(1-x)^{c-1}dx,$$

ergo habebimus

$$S=\frac{1}{C\int x^{n-1}(1-x)^{c-1}dx};$$

consequenter formula nostra generalis quaesita erit

$$T=\frac{\int x^{n-c-1}(1-x)^{c-1}dx}{\int x^{n-1}(1-x)^{c-1}dx}.$$

23. Quodsi ergo nostrae seriei per factores procedentis primum terminum ponamus $=A$, ipsa series erit

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ A, & \frac{1-cc}{1}A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}\cdot\frac{9-cc}{9}A \quad \text{etc.}; \end{array}$$

unde si sumamus $c=\frac{1}{2}$, erit haec series

$$A, \quad \frac{1\cdot 3}{2\cdot 2}A, \quad \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}A, \quad \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}A \quad \text{etc.},$$

cuius ergo terminus indici n respondens est

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}dx}{\int x^{n-1}(1-x)^{-\frac{1}{2}}dx},$$

qui posito $x=yy$ transit in hanc formam

$$\frac{\int y^{2n-2}(1-yy)^{-\frac{1}{2}}dy}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}dy},$$

unde patet terminum primum fore

$$A = \int \frac{dy}{\sqrt{(1-yy)}} : \int \frac{ydy}{\sqrt{(1-yy)}} = \frac{\pi}{2},$$

posito scilicet post integrationem $y = 1$.

PROBLEMA 3

24. *Denotet T terminum seriei indici n respondentem sintque T' et T'' termini sequentes pro indicibus $n+1$ et $n+2$; si proponatur inter ternos terminos se insequentes talis relatio, ut sit*

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'',$$

investigare formulam pro T, qua terminus generalis huius seriei exprimatur.

SOLUTIO

Assumatur pro T formula integralis $\int x^{n-1}dv$ huiusque integrale ita capiatur, ut evanescat posito $x = 0$, eruntque termini sequentes $T' = \int x^n dv$ et $T'' = \int x^{n+1}dv$, siquidem post integrationem variabili x certus valor determinatus tribuatur. Quamdiu autem haec quantitas x ut variabilis spectatur, ponamus esse

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'' + x^n Q$$

ac perspicuum est Q eiusmodi functionem esse debere ipsius x, quae evanescat, si loco x valor ille determinatus substituatur, quem autem a cyphra diversum esse oportet, quoniam iam assumsimus omnes istas formulas in nihilum abire posito $x = 0$. Quodsi vero absoluto calculo huic conditioni nullo modo satisfieri poterit, id erit indicio problema nostrum hac ratione resolvi non posse, ut scilicet eius terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1}dv$ exhibeatur.

25. Differentiemus nunc aequationem modo stabilitam ac divisione facta per x^{n-1} sequens prodibit aequatio

$$(\alpha n + a)dv = (\beta n + b)xdv + (\gamma n + c)xxdv + nQdx + xdQ,$$

quae, quia termini littera n affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes

$$1.\quad \alpha dv = \beta x dv + \gamma xx dv + Q dx,$$

$$2.\quad a dv = b x dv + c xx dv + x dQ,$$

ex quarum priore fit

$$dv = \frac{Q dx}{\alpha - \beta x - \gamma xx},$$

ex altera vero fit

$$dv = \frac{x dQ}{a - bx - cxx},$$

quorum valorum posterior per priorem divisus praebet

$$\frac{dQ}{Q} = \frac{dx(a - bx - cxx)}{x(\alpha - \beta x - \gamma xx)},$$

ex cuius ergo integratione valor ipsius Q elici debet, quo facto facile patebit, utrum is certo quodam casu praeter $x = 0$ evanescere possit. Imprimis autem hic notari convenit, si hoc integrale involvat huiusmodi factorem $e^{\frac{\lambda}{x}}$, tum solutionem quoque successu esse cariturам, quandoquidem posito $x = 0$ iste factor tantam involvet infiniti potestatem, ut, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

26. Quodsi igitur his conditionibus praescriptis satisfacere licuerit, tum invento valore litterae Q, quem ponamus fieri $= 0$ posito $x = f$, habebitur

$$dv = \frac{Q dx}{\alpha - \beta x - \gamma xx}$$

et formula generalis naturam seriei complectens erit

$$T = \int x^{n-1} dv = \int \frac{x^{n-1} Q dx}{\alpha - \beta x - \gamma xx},$$

quippe cuius integrale a termino $x = 0$ usque ad terminum $x = f$ extensum praebebit valorem termini T indici cuicunque n respondentis.

SCHOLION

27. Inventa autem tali relatione inter ternos terminos cuiuspiam seriei sibi invicem succedentes inde more solito formari poterit fractio continua, cuius valorem assignare licebit. Si enim characteres

$$T',\ T'',\ T''',\ T'''' \text{ etc.}$$

denotent ordine omnes terminos post T sequentes in infinitum, ex relationibus, quas inter se tenent, sequentes formulae deducentur. Ex relatione

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T''$$

deducitur

$$(\alpha n+a)\frac{T}{T'}=\beta n+b+\frac{(\gamma n+c)(\alpha n+\alpha+a)}{(\alpha n+\alpha+a)T':T''}.$$

Ex relatione sequente

$$(\alpha n+\alpha+a)T'=(\beta n+\beta+b)T''+(\gamma n+\gamma+c)T'''$$

deducitur

$$(\alpha n+\alpha+a)\frac{T'}{T''}=\beta n+\beta+b+\frac{(\gamma n+\gamma+c)(\alpha n+2\alpha+a)}{(\alpha n+2\alpha+a)T'':T'''}.$$

Simili modo sequentes relationes suppeditabunt

$$(\alpha n+2\alpha+a)\frac{T''}{T'''}=\beta n+2\beta+b+\frac{(\gamma n+2\gamma+c)(\alpha n+3\alpha+a)}{(\alpha n+3\alpha+a)T''':T''''},$$

$$(\alpha n+3\alpha+a)\frac{T'''}{T''''}=\beta n+3\beta+b+\frac{(\gamma n+3\gamma+c)(\alpha n+4\alpha+a)}{(\alpha n+4\alpha+a)T'''':T'''''};$$

unde manifestum est, si in prima formula continuo sequentes valores ordine substituantur, prodituram esse fractionem continuam, cuius valor aequalis erit formulae $(\alpha n+a)\frac{T}{T'}$.

28. Quodsi ergo loco n successive scribamus numeros 1, 2, 3, 4 etc., sequens problema circa fractiones continuas resolvere poterimus.

PROBLEMA 4

Proposita fractione continua huius formae

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{ etc.}}}}}}$$

eius valorem investigare.

SOLUTIO

Consideretur in genere ista relatio inter ternas quantitates sibi succedentes T, T', T'', quae sit

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T'',$$

atque ex praecedente problemate quaeratur valor ipsius T, siquidem fieri potest, hoc modo expressus

$$T=\int x^{n-1}dv=\int\frac{x^{n-1}Q\,dx}{\alpha-\beta x-\gamma xx},$$

cuius integrale ab $x=0$ usque ad $x=f$ extendatur, qua formula inventa ponatur

$$\int\frac{Q\,dx}{\alpha-\beta x-\gamma xx}=A \quad\text{et}\quad \int\frac{xQ\,dx}{\alpha-\beta x-\gamma xx}=B,$$

ita ut A et B sint valores ipsius T pro casibus $n=1$ et $n=2$; quibus definitis fractionis continuae propositae valor per praecedentia erit $=\frac{(\alpha+a)A}{B}$. Hanc igitur investigationem ad sequentia exempla accommodemus.

EXEMPLUM 1

29. *Investigare valorem fractionis continuae notissimae, quam olim* BROUNCKERUS[1]) *pro quadratura circuli protulit, quae est*

$$2+\cfrac{1\cdot 1}{2+\cfrac{3\cdot 3}{2+\cfrac{5\cdot 5}{2+\text{ etc.}}}}$$

1) Hanc celebrem fractionem continuam BROUNCKERUS epistola cum WALLISIO communicaverat. Vide I. WALLISII *Arithmeticam infinitorum,* Oxonii 1656, p. 182; *Opera*, t. I, Oxoniae

Quia omnes partes integrae laevam respicientes sunt constantes $=2$, pro nostra forma generali fiet

$$\beta+b=2, \quad 2\beta+b=2, \quad 3\beta+b=2 \quad \text{etc.};$$

erit ergo $\beta=0$ et $b=2$; at pro numeratoribus sequentium fractionum, quandoquidem constant binis factoribus, erit pro factoribus prioribus

$$\gamma+c=1, \quad 2\gamma+c=3, \quad 3\gamma+c=5, \quad 4\gamma+c=7 \quad \text{etc.},$$

unde concluditur $\gamma=2$ et $c=-1$, pro alteris vero erit

$$2\alpha+a=1, \quad 3\alpha+a=3, \quad 4\alpha+a=5 \quad \text{etc.},$$

unde $\alpha=2$ et $a=-3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{dQ}{Q}=-\frac{dx(3+2x-xx)}{2x(1-xx)},$$

quae per $1+x$ depressa praebet

$$\frac{dQ}{Q}=-\frac{dx(3-x)}{2x(1-x)},$$

unde integrando fit

$$lQ=-\frac{3}{2}lx+l(1-x) \quad \text{et hinc} \quad Q=\frac{1-x}{x^{\frac{3}{2}}},$$

ex quo valore porro sequitur

$$A=\int\frac{(1-x)dx}{2x^{\frac{3}{2}}(1-xx)}=\int\frac{dx}{2x(1+x)\sqrt{x}},$$

$$B=\int\frac{(1-x)dx}{2x^{\frac{1}{2}}(1-xx)}=\int\frac{dx}{2(1+x)\sqrt{x}}.$$

1695, p. 469. Notandum est fractionem BROUNCKERIANAM apud WALLISIUM hoc modo scriptam

$$1\frac{1}{2}\frac{9}{2}\frac{25}{2}\frac{49}{2}\frac{81}{2} \text{ etc.}$$

valorem fractionis $\frac{4}{\pi}$ repraesentare ideoque a fractione huius exempli unitate differre.

Vide etiam L. EULERI *Introductionem in analysin infinitorum,* Lausannae 1748, t. I, p. 305; *LEONHARDI EULERI Opera omnia,* series I, vol. 8. A. G.

30. In his autem valoribus istud incommodum deprehenditur, quod prius integrale evanescens reddi nequit posito $x = 0$. Hoc autem incommodum facile removeri potest, si fractionem continuam supremo membro truncemus et quaeramus valorem istius fractionis

$$2 + \cfrac{3 \cdot 3}{2 + \cfrac{5 \cdot 5}{2 + \text{etc.};}}$$

qui si repertus fuerit $= s$, erit ipsius propositae valor $= b + \frac{1}{s}$. Nunc vero comparatione instituta fit quidem ut ante $\beta = 0$ et $b = 2$, tum vero $\gamma = 2$ et $c = +1$, $\alpha = 2$ et $a = -1$, unde sequitur

$$\frac{dQ}{Q} = -\frac{dx(1+2x+xx)}{2x(1-xx)} = -\frac{dx(1+x)}{2x(1-x)},$$

unde integrando fit

$$lQ = -\frac{1}{2}lx + l(1-x) \quad \text{ideoque} \quad Q = \frac{1-x}{\sqrt{x}},$$

ex quo valore iam habebimus

$$A = \int \frac{(1-x)dx}{2(1-xx)\sqrt{x}} = \frac{1}{2}\int \frac{dx}{(1+x)\sqrt{x}}$$

et

$$B = \frac{1}{2}\int \frac{dx\sqrt{x}}{1+x};$$

ubi cum sit $Q = \frac{1-x}{\sqrt{x}}$, eius valor manifesto evanescit posito $x = 1$, quamobrem illa integralia a termino $x = 0$ usque ad $x = 1$ sunt extendenda.

31. Quo nunc haec integralia facilius eruamus, statuamus $x = zz$, ita ut termini integrationis etiamnunc sint $z = 0$ et $z = 1$, eritque

$$A = \int \frac{dz}{1+zz} = \text{A tang.}\, z = \frac{\pi}{4}$$

et

$$B = \int \frac{zzdz}{1+zz} = 1 - \frac{\pi}{4}$$

sicque habebimus $s = \frac{\pi}{4-\pi}$, quocirca ipsius fractionis Brounckerianae valor est $1 + \frac{4}{\pi}$, omnino uti olim Brounckerus iam invenerat.[1])

1) Sed vide notam p. 227. A. G.

EXEMPLUM 2

31[a].[1]). *Investigare valorem huius fractionis continuae* BROUNCKERIANAE[2]) *latius patentis*

$$b+\cfrac{1\cdot 1}{b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.}}}}$$

Ut hic incommodum superius evitemus, omittamus membrum supremum et quaeramus

$$s=b+\cfrac{3\cdot 3}{b+\cfrac{5\cdot 5}{b+\text{etc.},}}$$

quandoquidem tum erit valor quaesitus $=b+\frac{1}{s}$. Nunc igitur erit $\beta=0$ et $b=b$, $\gamma=2$, $c=1$, $\alpha=2$ et $a=-1$, unde fit

$$\frac{dQ}{Q}=-\frac{dx(1+bx+xx)}{2x(1-xx)}$$

ac proinde

$$lQ=-\frac{1}{2}lx-\frac{b-2}{4}l(1+x)+\frac{b+2}{4}l(1-x)$$

hincque

$$Q=\frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}},$$

1) In editione principe falso numerus 31 iteratur. A. G.

2) Num vero BROUNCKERUS ipse huiusmodi fractiones continuas casu $b \gtrless 2$ tractaverit, non liquet, cum investigationes eius nonnisi ex relatione a WALLISIO in propositione 191 *Arithmeticae infinitorum* (vide notam p. 227) data noverimus. Extant autem in fine huius propositionis haec WALLISII verba: „Atque hactenus Nobilissimi Viri mentem, quanta potui brevitate simul atque perspicuitate exposui; quaeque de ipsius methodo dicenda habui breviter indicavi". Vide I. WALLIS, *Opera*, t. I, p. 476. Vide etiam, id quod EULERUS ipse ad hanc quaestionem scripsit in Commentatione 123 (indicis ENESTROEMIANI): *De fractionibus continuis observationes*, Comment. acad. sc. Petrop. 11 (1739) 1750, p. 32, imprimis p. 39—42; *LEONHARDI EULERI Opera omnia*, series I, vol. 14.

Ceterum ne WALLISIUS quidem alios casus nisi $b=4n+2$ et $b=4n$ tractavit. Vide *Arithmeticam infinitorum*, p. 182—183; *Opera*, t. I, p. 470. A. G.

quae formula manifesto fit $=0$ ponendo $x=1$, siquidem $b+2$ fuerit numerus positivus, unde fit

$$dv = \frac{(1-x)^{\frac{b-2}{4}}\,dx}{2(1+x)^{\frac{b+2}{4}}\sqrt{x}}.$$

Hinc autem colligetur

$$A = \frac{1}{2}\int \frac{(1-x)^{\frac{b-2}{4}}\,dx}{(1+x)^{\frac{b+2}{4}}\sqrt{x}} \quad \text{et} \quad B = \frac{1}{2}\int \frac{(1-x)^{\frac{b-2}{4}}\,dx\sqrt{x}}{(1+x)^{\frac{b+2}{4}}}$$

sive ponendo $x=zz$ habebimus

$$A = \int \frac{(1-zz)^{\frac{b-2}{4}}\,dz}{(1+zz)^{\frac{b+2}{4}}} \quad \text{et} \quad B = \int \frac{(1-zz)^{\frac{b-2}{4}}\,zz\,dz}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z=0$ usque ad $z=1$ sunt extendenda. Ex his autem valoribus A et B erit $s=\frac{A}{B}$; ipsius igitur fractionis propositae valor erit $=b+\frac{1}{s}=b+\frac{B}{A}$.

32. Quodsi hic ponatur $b=2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares neque per logarithmos exprimere licet. Veluti si fuerit $b=4$, erit

$$A = \int \frac{dz\sqrt{(1-zz)}}{(1+zz)^{\frac{3}{2}}},$$

cuius valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis evadunt transcendentes, ita ut his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

33. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus huius formae

$$b = 4i + 2;$$

tum enim erit

$$A = \int \frac{(1-zz)^i dz}{(1+zz)^{i+1}} \quad \text{et} \quad B = \int \frac{(1-zz)^i zz dz}{(1+zz)^{i+1}};$$

quos ergo casus quomodo evolvi oporteat, operae pretium erit docere, quoniam WALLISIUS eos iam est contemplatus.[1])

34. Quoniam hoc negotium totum redit ad reductionem huiusmodi formularum integralium ad formas simpliciores, consideremus in genere formam $P = \frac{z^m}{(1+zz)^n}$, cuius differentiale sub sequentibus formis exhiberi potest:

$$1. \quad dP = \frac{mz^{m-1}dz}{(1+zz)^n} - \frac{2nz^{m+1}dz}{(1+zz)^{n+1}},$$

$$2. \quad dP = \frac{mz^{m-1}dz}{(1+zz)^{n+1}} - \frac{(2n-m)z^{m+1}dz}{(1+zz)^{n+1}},$$

$$3. \quad dP = -\frac{(2n-m)z^{m-1}dz}{(1+zz)^n} + \frac{2nz^{m-1}dz}{(1+zz)^{n+1}},$$

unde hanc triplicem reductionem integralium deducimus

$$\text{I.} \quad \int \frac{z^{m+1}dz}{(1+zz)^{n+1}} = \frac{m}{2n}\int \frac{z^{m-1}dz}{(1+zz)^n} - \frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n},$$

$$\text{II.} \quad \int \frac{z^{m+1}dz}{(1+zz)^{n+1}} = \frac{m}{2n-m}\int \frac{z^{m-1}dz}{(1+zz)^{n+1}} - \frac{1}{2n-m}\cdot\frac{z^m}{(1+zz)^n},$$

$$\text{III.} \quad \int \frac{z^{m-1}dz}{(1+zz)^{n+1}} = \frac{2n-m}{2n}\int \frac{z^{m-1}dz}{(1+zz)^n} + \frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n},$$

quarum reductionum ope casibus $b = 4i+2$ totum negotium absolvi et ad formulam $\frac{\pi}{4}$ reduci poterit, siquidem post integrationem sumatur $z = 1$.

35. Sit $i = 1$ ideoque $b = 6$ eritque

$$A = \int \frac{(1-zz)dz}{(1+zz)^2} \quad \text{et} \quad B = \int \frac{(1-zz)zzdz}{(1+zz)^2}.$$

Nunc igitur reperiemus per reductionem tertiam

$$\int \frac{dz}{(1+zz)^2} = \frac{1}{2}\int \frac{dz}{1+zz} + \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4}$$

1) Vide notam 2 p. 230. A. G.

et per reductionem primam

$$\int\frac{zzdz}{(1+zz)^2}=\frac{1}{2}\int\frac{dz}{1+zz}-\frac{1}{2}\cdot\frac{z}{1+zz}=\frac{\pi}{8}-\frac{1}{4},$$

porro

$$\int\frac{z^4dz}{(1+zz)^2}=\frac{3}{2}\int\frac{zzdz}{1+zz}-\frac{1}{2}\cdot\frac{z^3}{1+zz}=\frac{5}{4}-\frac{3\pi}{8}.$$

Ex his iam valoribus colligitur $A=\frac{1}{2}$ et $B=\frac{\pi}{2}-\frac{3}{2}$ ideoque $\frac{B}{A}=\pi-3$, quocirca orietur ista summatio

$$3+\pi=6+\cfrac{1\cdot 1}{6+\cfrac{3\cdot 3}{6+\cfrac{5\cdot 5}{6+\cfrac{7\cdot 7}{6+\text{etc.}}}}}$$

36. Sit nunc $i=2$ et $b=10$ eritque

$$A=\int\frac{(1-zz)^2dz}{(1+zz)^3}\quad\text{et}\quad B=\int\frac{zz(1-zz)^2dz}{(1+zz)^3}.$$

Quo harum integralium valores investigemus, sequentes evolvamus formulas

$$\int\frac{dz}{(1+zz)^3}=\frac{3}{4}\int\frac{dz}{(1+zz)^2}+\frac{1}{4}\cdot\frac{z}{(1+zz)^2}=\frac{3\pi}{32}+\frac{1}{4},$$

$$\int\frac{zzdz}{(1+zz)^3}=\frac{1}{4}\int\frac{dz}{(1+zz)^2}-\frac{1}{4}\cdot\frac{z}{(1+zz)^2}=\frac{\pi}{32},$$

$$\int\frac{z^4dz}{(1+zz)^3}=\frac{3}{4}\int\frac{zzdz}{(1+zz)^2}-\frac{1}{4}\cdot\frac{z^3}{(1+zz)^2}=\frac{3\pi}{32}-\frac{1}{4},$$

$$\int\frac{z^6dz}{(1+zz)^3}=\frac{5}{4}\int\frac{z^4dz}{(1+zz)^2}-\frac{1}{4}\cdot\frac{z^5}{(1+zz)^2}=\frac{3}{2}-\frac{15\pi}{32}.$$

Ex quibus iam valoribus deducitur $A=\frac{\pi}{8}$ et $B=2-\frac{5\pi}{8}$ ideoque $\frac{B}{A}=\frac{16-5\pi}{\pi}$, unde emergit sequens summatio

$$\frac{5\pi+16}{\pi}=10+\cfrac{1\cdot 1}{10+\cfrac{3\cdot 3}{10+\cfrac{5\cdot 5}{10+\text{etc.}}}}$$

37. Si b esset numerus negativus, investigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-c + \cfrac{\gamma}{-d + \cfrac{\delta}{-e + \text{etc.},}}}}$$

semper erit

$$-s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \text{etc.},}}}}$$

unde, si habeatur valor istius expressionis, idem negative sumtus dabit valorem illius.

EXEMPLUM 3

38. *Proposita sit fractio continua, cuius valorem investigari oporteat, ista*

$$1 + \cfrac{1 \cdot 1}{3 + \cfrac{3 \cdot 3}{5 + \cfrac{5 \cdot 5}{7 + \cfrac{7 \cdot 7}{9 + \text{etc.}}}}}$$

Quo fractiones supra [§ 28] allegatas [adhibeamus,] omisso membro supremo sit

$$s = 3 + \cfrac{3 \cdot 3}{5 + \cfrac{5 \cdot 5}{7 + \cfrac{7 \cdot 7}{9 + \text{etc.}}}}$$

eritque $\beta + b = 3$, $2\beta + b = 5$ ideoque $\beta = 2$ et $b = 1$, tum vero ut ante $\alpha = 2$, $a = -1$, $\gamma = 2$ et $c = +1$; invento autem s erit valor quaesitus $= 1 + \frac{1}{s}$. Nunc igitur habebimus

$$\frac{dQ}{Q} = -\frac{dx(1 + x + xx)}{2x(1 - x - xx)}.$$

Est vero

$$\frac{1+x+xx}{x(1-x-xx)}=\frac{1}{x}+\frac{2+2x}{1-x-xx},$$

unde fit

$$lQ=-\frac{1}{2}lx-\int\frac{dx(1+x)}{1-x-xx}.$$

Porro vero pro formula $\int\frac{dx(1+x)}{1-x-xx}$ invenienda statuamus denominatorem

$$1-x-xx=(1-fx)(1-gx)$$

eritque $f+g=1$ et $fg=-1$, unde fit

$$f=\frac{1+\sqrt{5}}{2}\quad\text{et}\quad g=\frac{1-\sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1+x}{1-x-xx}=\frac{\mathfrak{A}}{1-fx}+\frac{\mathfrak{B}}{1-gx},$$

unde reperietur

$$\mathfrak{A}=\frac{1+f}{f-g}\quad\text{et}\quad\mathfrak{B}=-\frac{1+g}{f-g},$$

sive substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A}=\frac{\sqrt{5}+3}{2\sqrt{5}}\quad\text{et}\quad\mathfrak{B}=\frac{\sqrt{5}-3}{2\sqrt{5}},$$

quibus inventis erit

$$\int\frac{dx(1+x)}{1-x-xx}=-\frac{\mathfrak{A}}{f}l(1-fx)-\frac{\mathfrak{B}}{g}l(1-gx)$$

$$=-\frac{1+\sqrt{5}}{2\sqrt{5}}l(1-fx)-\frac{\sqrt{5}-1}{2\sqrt{5}}l(1-gx),$$

quocirca fiet

$$lQ=-\frac{1}{2}lx+\frac{\sqrt{5}+1}{2\sqrt{5}}l(1-fx)+\frac{\sqrt{5}-1}{2\sqrt{5}}l(1-gx),$$

consequenter

$$Q=\frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}}(1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

qui valor duobus casibus evanescit, altero, quo

$$x = \frac{1}{f} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2},$$

altero vero, quo

$$x = \frac{1}{g} = -\frac{1+\sqrt{5}}{2};$$

utrovis autem utamur, res eodem redibit.

39. Ex hoc autem valore habebimus

$$A = \int \frac{Qdx}{1-x-xx} \quad \text{et} \quad B = \int \frac{Qxdx}{1-x-xx},$$

unde porro deducitur

$$s = (\alpha + a)\frac{A}{B} = \frac{A}{B};$$

hinc propositae fractionis summa erit $1 + \frac{B}{A}$. Hinc autem nihil ulterius concludere licet ob formulas differentiales non solum irrationales, sed etiam vere transcendentes ob exponentes surdos.

EXEMPLUM 4

40. *Proposita sit haec fractio continua*

$$b + \cfrac{1\cdot 1}{b + \cfrac{2\cdot 2}{b + \cfrac{3\cdot 3}{b + \cfrac{4\cdot 4}{b + \text{etc.},}}}}$$

ubi est $\beta = 0$, $b = b$.

Nunc consideremus hanc formam

$$s = b + \cfrac{2\cdot 2}{b + \cfrac{3\cdot 3}{b + \text{etc.},}}$$

quippe quo valore invento quaesitus erit $= b + \frac{1}{s}$. Habebimus igitur

$\gamma + c = 2$, $2\gamma + c = 3$ ideoque $\gamma = 1$ et $c = 1$, deinde erit $\alpha = \gamma = 1$, $a = 0$ et $c = 1$. Hinc igitur colligimus

$$\frac{dQ}{Q} = -\frac{dx(bx+xx)}{x(1-xx)} = -\frac{dx(b+x)}{1-xx}$$

ideoque

$$lQ = -\frac{b}{2}l\frac{1+x}{1-x} + \frac{1}{2}l(1-xx)$$

hincque

$$Q = \frac{(1-x)^{\frac{b}{2}}\sqrt{(1-xx)}}{(1+x)^{\frac{b}{2}}} = \frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto evanescit posito $x = 1$. Hinc igitur fiet

$$A = \int\frac{Qdx}{1-xx} = \int\frac{(1-x)^{\frac{b+1}{2}}dx}{(1+x)^{\frac{b-1}{2}}(1-xx)} = \int\frac{(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}}$$

et

$$B = \int\frac{x(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}},$$

tum autem erit $s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$ ideoque summa quaesita $b + \frac{B}{A}$.

41. Percurramus nunc casus praecipuos ac primo sit $b = 1$ eritque

$$A = \int\frac{dx}{1+x} = l(1+x) = l2 \quad \text{et} \quad B = \int\frac{xdx}{1+x} = x - \int\frac{dx}{1+x} = 1 - l2$$

ideoque $b + \frac{B}{A} = \frac{1}{l2}$; ergo hinc prodibit ista summatio

$$\frac{1}{l2} = 1 + \cfrac{1\cdot 1}{1 + \cfrac{2\cdot 2}{1 + \cfrac{3\cdot 3}{1 + \text{etc.}}}}$$

42. Sit nunc $b = 2$ eritque

$$A = \int\frac{dx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}} \quad \text{et} \quad B = \int\frac{xdx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas statuamus

$$\frac{\sqrt{(1-x)}}{\sqrt{(1+x)}}=z$$

eritque $x=\frac{1-zz}{1+zz}$, unde terminis integrationis $x=0$ et $x=1$ respondebunt $z=1$ et $z=0$; tum vero erit

$$1+x=\frac{2}{1+zz} \quad \text{et} \quad dx=-\frac{4z\,dz}{(1+zz)^2}$$

hincque colligitur

$$A=-2\int\frac{zz\,dz}{1+zz}=-2z+2\,\mathrm{A\ tang.}\,z=2-\frac{\pi}{2},$$

porro fit

$$B=-2\int\frac{zz\,dz}{(1+zz)^2}+2\int\frac{z^4\,dz}{(1+zz)^2}.$$

Per reductiones igitur supra (§ 35) monstratas, si hic scilicet terminos integrationis $z=1$ et $z=0$ permutemus, ut habeamus

$$B=+2\int\frac{zz\,dz}{(1+zz)^2}-2\int\frac{z^4\,dz}{(1+zz)^2},$$

erit

$$B=2\left(\frac{\pi}{8}-\frac{1}{4}\right)-2\left(\frac{5}{4}-\frac{3\pi}{8}\right)=\pi-3,$$

unde sequitur ista summatio

$$\frac{2}{4-\pi}=2+\cfrac{1\cdot 1}{2+\cfrac{2\cdot 2}{2+\cfrac{3\cdot 3}{2+\cfrac{4\cdot 4}{2+\text{etc.},}}}}$$

quae Brounckerianae simplicitate nihil cedit.

43. Si ponamus $b=0$, fractio continua abit in sequens continuum productum

$$\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 3}{4\cdot 4}\cdot\frac{5\cdot 5}{6\cdot 6}\cdot\frac{7\cdot 7}{8\cdot 8}\cdot\text{etc.};$$

hoc autem casu fit

$$A = \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \quad \text{et} \quad B = \int \frac{x\,dx}{\sqrt{(1-xx)}} = 1,$$

unde istius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum iam dudum cognitis, quandoquidem hoc productum est ipsa progressio WALLISIANA.[1])

EXEMPLUM 5

44. *Proposita sit haec fractio continua, ubi* $\beta = 0$, $b = b$ *et numeratores numeri trigonales,*

$$b + \cfrac{1}{b + \cfrac{3}{b + \cfrac{6}{b + \cfrac{10}{b + \text{etc.}}}}}$$

Omisso supremo membro statuamus

$$s = b + \cfrac{3}{b + \cfrac{6}{b + \text{etc.}}}$$

et primo numeratores per producta repraesentemus hoc modo

$$3 = 2 \cdot \frac{3}{2}, \quad 6 = 3 \cdot \frac{4}{2}, \quad 10 = 4 \cdot \frac{5}{2},$$

quorum priores comparentur cum formulis $\gamma + c$, $2\gamma + c$, $3\gamma + c$, posteriores vero cum formulis $2\alpha + a$, $3\alpha + a$, $4\alpha + a$, eritque $\gamma = 1$, $c = 1$, $\alpha = \frac{1}{2}$, $a = \frac{1}{2}$, unde erit

$$\frac{dQ}{Q} = \frac{dx(\frac{1}{2} - bx - xx)}{x(\frac{1}{2} - xx)} = \frac{dx(1 - 2bx - 2xx)}{x(1 - 2xx)}$$

sive

$$\frac{dQ}{Q} = \frac{dx}{x} - \frac{2b\,dx}{1 - 2xx},$$

---

1) Notandum autem est progressionem WALLISIANAM hanc formam habere

$$\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot \text{etc.}}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot \text{etc.}} \quad \text{seu} \quad \frac{9 \cdot 25 \cdot 49 \cdot 81 \cdot \text{etc.}}{8 \cdot 24 \cdot 48 \cdot 80 \cdot \text{etc.}}$$

neque valorem $\frac{2}{\pi}$ sed $\frac{4}{\pi}$ repraesentare. Vide WALLISII *Arithmeticam infinitorum*, p. 179; *Opera*, t. I, p. 467—469. A. G.

cuius integrale est

$$lQ = lx - \frac{b}{\sqrt{2}}\, l\frac{1+x\sqrt{2}}{1-x\sqrt{2}},$$

ergo

$$Q = \frac{x(1-x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1+x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

quae formula evanescit casu $x = \frac{1}{\sqrt{2}}$. Hinc igitur erit

$$dv = \frac{2x(1-x\sqrt{2})^{\frac{b}{\sqrt{2}}}dx}{(1-2xx)(1+x\sqrt{2})^{\frac{b}{\sqrt{2}}}}.$$

Sit $\frac{b}{\sqrt{2}} = \lambda$ eritque

$$A = 2\int\frac{x(1-x\sqrt{2})^{\lambda}dx}{(1-2xx)(1+x\sqrt{2})^{\lambda}} = 2\int\frac{x(1-x\sqrt{2})^{\lambda-1}dx}{(1+x\sqrt{2})^{\lambda+1}}$$

et

$$B = 2\int\frac{xx(1-x\sqrt{2})^{\lambda-1}dx}{(1+x\sqrt{2})^{\lambda+1}},$$

ubi post integrationem statuitur $x = \frac{1}{\sqrt{2}}$; tum autem fit $s = \frac{A}{B}$ hincque valor fractionis propositae $= b + \frac{B}{A}$.

45. Nisi igitur fuerit $\lambda = \frac{b}{\sqrt{2}}$ numerus rationalis, hos valores commode assignare non licet. Sit igitur $b = \sqrt{2}$ sive $\lambda = 1$ eritque

$$A = 2\int\frac{xdx}{(1+x\sqrt{2})^2} \quad \text{et} \quad B = 2\int\frac{xxdx}{(1+x\sqrt{2})^2}.$$

Hinc integrando colligitur

$$A = l(1+x\sqrt{2}) - \frac{x\sqrt{2}}{1+x\sqrt{2}}$$

ideoque posito $x\sqrt{2} = 1$ fiet $A = l2 - \frac{1}{2}$; tum vero reperitur

$$B = \frac{3}{2\sqrt{2}} - \sqrt{2}\cdot l2,$$

quare ob $b = \sqrt{2}$ erit $b + \frac{B}{A} = \frac{1}{\sqrt{2(2l2-1)}}$, unde sequitur haec summatio

$$\frac{1}{\sqrt{2(2l2-1)}} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

SCHOLION

46. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, huiusmodi formam habere solent

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}}$$

ubi omnes numeratores sunt unitates, denominatores vero a, b, c, d, e etc. numeri integri. Verum ope nostrae methodi difficulter talium formarum valores eruere licet, etiamsi numeri a, b, c, d, e progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

EXEMPLUM

47. *Proposita sit ista fractio continua*

$$\beta + b + \cfrac{1}{2\beta + b + \cfrac{1}{3\beta + b + \cfrac{1}{4\beta + b + \cfrac{1}{5\beta + b + \text{etc.},}}}}$$

ubi $\alpha = 0,\ \gamma = 0,\ a = 1,\ c = 1.$

Hinc fit

$$\frac{dQ}{Q} = -\frac{dx(1 - bx - xx)}{\beta xx},$$

unde

$$lQ = \frac{1}{\beta x} + \frac{b}{\beta} lx + \frac{x}{\beta} \quad \text{et} \quad Q = e^{\frac{1+xx}{\beta x}} x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu evanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positivus. Verum si pro β sumamus numeros negativos, puta $\beta = -m$, tum valor $Q = x^{\frac{-b}{m}} e^{\frac{-(1+xx)}{mx}}$ manifesto evanescit, tam si $x = 0$ quam si $x = \infty$. Hinc autem erit

$$dv = \frac{x^{\frac{-b}{m}} e^{\frac{-1-xx}{mx}} dx}{mxx},$$

quamobrem habebimus

$$A = \frac{1}{m}\int \frac{dx}{x^{2+\frac{b}{m}} e^{\frac{1+xx}{mx}}} \quad \text{et} \quad B = \frac{1}{m}\int \frac{dx}{x^{1+\frac{b}{m}} e^{\frac{1+xx}{mx}}}.$$

His valoribus inventis formula $\frac{A}{B}$ exprimet summam huius fractionis continuae

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

quamobrem formula illa negative sumta $-\frac{A}{B}$ exprimet valorem huius fractionis continuae

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x = 0$ ad $x = \infty$ extendi possent. Verum istae formulae ita sunt comparatae, ut earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quominus fractio $\frac{A}{B}$ valores satis cognitos involvere queat, etiamsi eos nullo adhuc modo assignare valeamus.

48[1]). Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet:

1) In editione principe huic paragrapho falso numerus 49 inscribitur. A. G.

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}}=\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}\ ^{1)}$$

et

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}}=\cot.\frac{1}{n}.$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n$, $2m-b=3n$ ideoque $m=2n$ et $b=n$, unde fit

$$A=\frac{1}{2n}\int\frac{dx}{x^{\frac{5}{2}}e^{\frac{1+xx}{2nx}}}\quad\text{et}\quad B=\frac{1}{2n}\int\frac{dx}{x^{\frac{3}{2}}e^{\frac{1+xx}{2nx}}},$$

unde iam discimus, si hae duae formulae integrentur a termino $x=0$ usque ad terminum $x=\infty$, tum fore

$$\frac{A}{B}=\frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet hanc convenientiam demonstrandi.

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1) Editio princeps (atque etiam editiones 594a, 594b, 594A indicis ENESTROEMIANI): $\frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1}$. Correctam autem formulam EULERUS ipse iam dederat in Commentatione 71 (indicis ENESTROEMIANI): *De fractionibus continuis,* Comment. acad. sc. Petrop. 9 (1737), 1744, p. 98 (vide ibi secundam formulam § 30, ubi sufficit loco s scribere $\frac{n}{2}$); *LEONHARDI EULERI Opera omnia,* series I, vol. 14. Vide etiam EULERI Commentationem 595 (indicis ENESTROEMIANI): *Summatio fractionis continuae, cuius indices progressionem arithmeticam constituunt, dum numeratores omnes sunt unitates; ubi simul resolutio aequationis RICCATIANAE per huiusmodi fractiones docetur, Opuscula analytica* 2, 1785, p. 217; *LEONHARDI EULERI Opera omnia,* series I, vol. 23. Statim in § 1 huius Commentationis 595 invenitur correcta formula $\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$ atque praeterea etiam formula pro $\cot.\frac{1}{n}$, „quarum quidem altera ex altera facile deducitur, si loco n scribatur $n\sqrt{-1}$". A. G.

SPECULATIONES SUPER FORMULA INTEGRALI
$$\int \frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}$$
UBI SIMUL EGREGIAE OBSERVATIONES CIRCA FRACTIONES CONTINUAS OCCURRUNT

Commentatio 606 indicis ENESTROEMIANI
Acta academiae scientiarum Petropolitanae 1782: II, 1786, p. 62—84

1. Incipiamus a casu simplicissimo, quo $n=0$, et quaeramus integrale formulae

$$\frac{dx}{\sqrt{(aa-2bx+cxx)}},$$

quae posito $x=\frac{b+z}{c}$ transit in hanc

$$\frac{dz}{\sqrt{(aacc-bbc+czz)}},$$

ubi duo casus distingui convenit, prout c fuerit vel quantitas positiva vel negativa.

Sit igitur primo $c=+ff$ et formula nostra fiet

$$\frac{dz}{f\sqrt{(aaff-bb+zz)}},$$

cuius integrale est

$$\frac{1}{f}l\frac{z+\sqrt{(aaff-bb+zz)}}{C},$$

ideoque erit nostrum integrale

$$\frac{1}{\sqrt{c}}\, l\, \frac{cx-b+\sqrt{(aac-2bcx+ccxx)}}{C},$$

quod ergo ita sumtum, ut evanescat posito $x=0$, evadet

$$\frac{1}{\sqrt{c}}\, l\, \frac{cx-b+\sqrt{c(aa-2bx+cxx)}}{-b+a\sqrt{c}}.$$

At vero si c fuerit quantitas negativa, puta $c=-gg$, formula differentialis per z expressa erit

$$\frac{dz}{g\sqrt{(aagg+bb-zz)}},$$

cuius integrale est

$$\frac{1}{g}\,\text{A sin.}\frac{z}{\sqrt{(aagg+bb)}}+C,$$

quare integrale ita sumtum, ut evanescat posito $x=0$, fiet

$$=\frac{1}{g}\,\text{A sin.}\frac{cx-b}{\sqrt{(aagg+bb)}}+\frac{1}{g}\,\text{A sin.}\frac{b}{\sqrt{(aagg+bb)}}.$$

2. Denotet nunc Π valorem formulae integralis $\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}$ ita sumtum, ut evanescat posito $x=0$, sive c fuerit quantitas positiva sive negativa; ac si sit $c=+ff$, erit, uti vidimus,

$$\Pi=\frac{1}{f}\, l\, \frac{ffx-b+f\sqrt{(aa-2bx+ffxx)}}{af-b};$$

altero vero casu, quo $c=-gg$, erit

$$\Pi=-\frac{1}{g}\,\text{A sin.}\frac{ggx+b}{\sqrt{(aagg+bb)}}+\frac{1}{g}\,\text{A sin.}\frac{b}{\sqrt{(aagg+bb)}}$$

sive ambobus arcubus contractis habebimus

$$\Pi=\frac{1}{g}\,\text{A sin.}\frac{bg\sqrt{(aa-2bx-ggxx)}-abg-ag^3x}{aagg+bb}.$$

Quoniam igitur mox ostendemus integrationem formulae generalis $\int \frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}$ semper reduci posse ad casum $n=0$, si modo fuerit n numerus integer positivus, omnia haec integralia per istum valorem Π exprimi poterunt.

3. Iam post integrationem quantitati variabili x eiusmodi valorem constantem tribuamus, quo formula irrationalis $\sqrt{(aa-2bx+cxx)}$ ad nihilum redigatur, id quod fit, si sumatur

$$x = \frac{b \pm \sqrt{(bb-aac)}}{c},$$

ideoque duobus casibus. Ponamus pro utroque casu functionem Π abire in Δ, ita ut casu $c = ff$ sit

$$\Delta = \frac{1}{f} l \frac{\sqrt{(bb-aaff)}}{af-b} = \frac{1}{f} l \sqrt{\frac{b+af}{b-af}},$$

pro altero autem casu, quo $c = -gg$,

$$\Delta = \frac{1}{g} \text{A sin.} \frac{\pm ag\sqrt{(bb+aagg)}}{aagg+bb} = \frac{1}{g} \text{A sin.} \frac{ag}{\sqrt{(bb+aagg)}}.$$

Hos autem valores Δ in sequentibus casibus, quibus ipsa formula radicalis $\sqrt{(aa-2bx+cxx)}$ evanescit, potissimum sumus contemplaturi.

4. Nunc ad sequentem casum progressuri consideremus formulam

$$s = \sqrt{(aa-2bx+cxx)} - a,$$

ut scilicet evanescat facto $x=0$, et quoniam est

$$ds = \frac{-bdx+cxdx}{\sqrt{(aa-2bx+cxx)}},$$

erit vicissim integrando

$$c\int \frac{xdx}{\sqrt{(aa-2bx+cxx)}} = b\int \frac{dx}{\sqrt{(aa-2bx+cxx)}} + s,$$

unde colligimus

$$\int \frac{xdx}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\Pi + \frac{\sqrt{(aa-2bx+cxx)}-a}{c};$$

quare si post integrationem statuamus $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, quippe quibus casibus fit $\sqrt{(aa-2bx+cxx)}=0$ et $\Pi=\Delta$, fiet

$$\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\Delta-\frac{a}{c}.$$

5. Sumamus porro

$$s=x\sqrt{(aa-2bx+cxx)};$$

fiet

$$ds=\frac{aadx-3bxdx+2cxxdx}{\sqrt{(aa-2bx+cxx)}},$$

unde vicissim integrando colligitur

$$2c\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}=3b\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}-aa\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}+s,$$

unde statim pro casu $\sqrt{(aa-2bx+cxx)}=0$ deducimus

$$\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}=\frac{3bb-aac}{2cc}\Delta-\frac{3ab}{2cc}.$$

6. Iam ad altiores potestates ascensuri statuamus

$$s=xx\sqrt{(aa-2bx+cxx)},$$

et quia hinc fit

$$ds=\frac{2aaxdx-5bxxdx+3cx^3dx}{\sqrt{(aa-2bx+cxx)}},$$

erit

$$3c\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}}=5b\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}}-2aa\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}+s$$

hincque porro pro casu, quo post integrationem statuitur $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, habebitur

$$\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}}=\frac{5b^3-3aabc}{2c^3}\Delta-\frac{15abb}{6c^3}+\frac{2a^3}{3cc}$$
$$=\left(\frac{5b^3}{2c^3}-\frac{3aab}{2cc}\right)\Delta-\frac{5abb}{2c^3}+\frac{2a^3}{3cc}.$$

7. Simili modo sit

$$s = x^3\sqrt{(aa-2bx+cxx)},$$

et quia hinc fit

$$ds = \frac{3aaxxdx - 7bx^3dx + 4cx^4dx}{\sqrt{(aa-2bx+cxx)}},$$

erit vicissim integrando

$$4c\int\frac{x^4dx}{\sqrt{(aa-2bx+cxx)}} = 7b\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}} - 3aa\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}} + s;$$

tum igitur pro casu, quo fit $\sqrt{(aa-2bx+cxx)} = 0$, habebimus

$$\int\frac{x^4dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{35b^4}{8c^4} - \frac{15aabb}{4c^3} + \frac{3a^4}{8cc}\right)\varDelta - \frac{35ab^3}{8c^4} + \frac{55a^3b}{24c^3}.$$

8. Quo autem ordo in his formulis melius explorari possit, singulas exhibeamus per factores, quemadmodum ordine oriuntur, sine ulla abbreviatione atque hoc modo formulae integrales inventae ita repraesententur:

$$\int\frac{dx}{\sqrt{(aa-2bx+cxx)}} = \varDelta,$$

$$\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\varDelta - \frac{a}{c},$$

$$\int\frac{xxdx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1\cdot 3bb}{1\cdot 2cc} - \frac{aa}{1\cdot 2c}\right)\varDelta - \frac{1\cdot 3ab}{1\cdot 2cc},$$

$$\int\frac{x^3dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1\cdot 3\cdot 5b^3}{1\cdot 2\cdot 3c^3} - \frac{1\cdot 3\cdot 3aab}{1\cdot 2\cdot 3cc}\right)\varDelta - \frac{1\cdot 3\cdot 5abb}{1\cdot 2\cdot 3c^3} + \frac{1\cdot 2\cdot 2a^3}{1\cdot 2\cdot 3cc},$$

$$\int\frac{x^4dx}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{1\cdot 3\cdot 5\cdot 7b^4}{1\cdot 2\cdot 3\cdot 4c^4} - \frac{1\cdot 3\cdot 5\cdot 6aabb}{1\cdot 2\cdot 3\cdot 4c^3} + \frac{1\cdot 3\cdot 3a^4}{1.2\cdot 3.4cc}\right)\varDelta$$
$$- \frac{1\cdot 3\cdot 5\cdot 7ab^3}{1\cdot 2\cdot 3\cdot 4c^4} + \frac{1\cdot 5\cdot 11a^3b}{1\cdot 2\cdot 3\cdot 4c^3}.$$

9. Instituamus nunc in genere istam evolutionem sumendo

$$s = x^n\sqrt{(aa-2bx+cxx)},$$

et quia hinc fit

$$ds = \frac{naax^{n-1}dx - (2n+1)bx^ndx + (n+1)cx^{n+1}dx}{\sqrt{(aa-2bx+cxx)}},$$

inde vicissim integrando colligitur

$$(n+1)c\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}}=(2n+1)b\int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}$$
$$-naa\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}}+x^n\sqrt{(aa-2bx+cxx)}.$$

Quodsi vero iam ante elicuerimus

$$\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}}=M\varDelta-\mathfrak{M} \quad \text{et} \quad \int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}=N\varDelta-\mathfrak{N},$$

ita ut hae duae formulae sint cognitae, sequens ex iis ita determinabitur, ut sit

$$\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{(2n+1)bN}{(n+1)c}-\frac{naaM}{(n+1)c}\right)\varDelta-\frac{(2n+1)b\mathfrak{N}}{(n+1)c}+\frac{naa\mathfrak{M}}{(n+1)c}.$$

Hoc igitur modo has integrationes, quousque libuerit, continuare licet, dum ex binis quibusque sequens ope huius regulae formatur, ita ut omnia haec integralia vel a logarithmis vel ab arcubus circularibus pendeant, prouti coefficiens c fuerit vel positivus vel negativus. Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positivus.

10. Ex forma integrali modo inventa, si post integrationem statuatur $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, unde fit $s=0$, erit

$$naa\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}}$$
$$=(2n+1)b\int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}-(n+1)c\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}};$$

unde si brevitatis gratia ponamus

$$\int\frac{x^{n-1}dx}{\sqrt{(aa-2bx+cxx)}}=P, \qquad \int\frac{x^n dx}{\sqrt{(aa-2bx+cxx)}}=Q,$$
$$\int\frac{x^{n+1}dx}{\sqrt{(aa-2bx+cxx)}}=R, \qquad \int\frac{x^{n+2}dx}{\sqrt{(aa-2bx+cxx)}}=S \text{ etc.},$$

hae quantitates P, Q, R, S etc. ita a se invicem pendent, ut sit

$$\begin{aligned}
naaP &= (2n+1)bQ-(n+1)cR,\\
(n+1)aaQ &= (2n+3)bR-(n+2)cS,\\
(n+2)aaR &= (2n+5)bS-(n+3)cT,\\
(n+3)aaS &= (2n+7)bT-(n+4)cU,\\
(n+4)aaT &= (2n+9)bU-(n+5)cW\\
&\text{etc.}
\end{aligned}$$

Ex his relationibus deducuntur sequentes determinationes

$$\begin{aligned}
\frac{P}{Q} &= \frac{(2n+1)b}{naa}-\frac{(n+1)c}{naaQ:R},\\
\frac{Q}{R} &= \frac{(2n+3)b}{(n+1)aa}-\frac{(n+2)c}{(n+1)aaR:S},\\
\frac{R}{S} &= \frac{(2n+5)b}{(n+2)aa}-\frac{(n+3)c}{(n+2)aaS:T},\\
\frac{S}{T} &= \frac{(2n+7)b}{(n+3)aa}-\frac{(n+4)c}{(n+3)aaT:U}\\
&\text{etc.};
\end{aligned}$$

hinc igitur patet singulas has fractiones $\frac{P}{Q}$, $\frac{Q}{R}$, $\frac{R}{S}$ etc. per sequentes satis commode determinari.

11. Quodsi iam in qualibet harum expressionum valores modo exhibiti successive substituantur, pro fractione $\frac{P}{Q}$ impetrabimus fractionem continuam in infinitum progredientem, quae erit

$$naa\frac{P}{Q}=(2n+1)b-\cfrac{(n+1)^2aac}{(2n+3)b-\cfrac{(n+2)^2aac}{(2n+5)b-\cfrac{(n+3)^2aac}{(2n+7)b-\cfrac{(n+4)^2aac}{(2n+9)b-\text{etc.}}}}}$$

sicque pervenimus ad fractionem continuam satis concinnam et ordine perspicuo progredientem, cuius igitur valor semper vel per logarithmos (si fuerit $c>0$) vel per arcus circulares (si fuerit $c<0$) exprimi potest.

12. Sumamus nunc $n=1$ ac fiet

$$P=\int\frac{dx}{\sqrt{(aa-2bx+cxx)}}=\varDelta$$

et

$$Q=\int\frac{xdx}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\varDelta-\frac{a}{c},$$

qui casus nobis suppeditat sequentem fractionem continuam

$$\frac{aac\varDelta}{b\varDelta-a}=3b-\cfrac{4aac}{5b-\cfrac{9aac}{7b-\cfrac{16aac}{9b-\cfrac{25aac}{11b-\text{etc.}}}}}$$

quae ob elegantiam omni attentione digna est censenda. Hic autem notasse iuvabit, si c fuerit numerus negativus, tum omnes numeratores in hac fractione evadere positivos.

12[a][1]. Fractio autem haec continua capite quasi trunca videtur; unde si superne ei adiungatur membrum $b-aac$, ea adhuc concinnior eiusque valor simplicior reddetur. Si enim ista fractio brevitatis gratia designetur littera S, ita ut sit $S=\frac{aac\varDelta}{b\varDelta-a}$, adiecto isto membro eius valor erit $b-\frac{aac}{S}=\frac{a}{\varDelta}$ sicque habebimus

$$\frac{a}{\varDelta}=b-\cfrac{aac}{3b-\cfrac{4aac}{5b-\cfrac{9aac}{7b-\cfrac{16aac}{9b-\cfrac{25aac}{11b-\text{etc.}}}}}}$$

quae expressio eo magis est memorabilis, quod nulla adhuc via patet, qua talis fractionis continuae valor a priori inveniri potest.

1) In editione principe falso numerus 12 iteratur. A. L.

13. Evolvamus nunc seorsim binos casus supra memoratos, et quos sollicite a se invicem distingui convenit. Sit igitur primo $c=ff$ atque supra invenimus fore

$$\varDelta=\frac{1}{f}\,l\,\frac{\sqrt{(bb-aaff)}}{af-b},$$

ubi signum radicale ambigue accipi potest. Ante omnia igitur necesse est, ut sit $bb>aaff$, quia alioquin haec expressio evaderet imaginaria; duo ergo casus se offerunt, prouti b fuerit quantitas sive positiva sive negativa.

Priore casu, quo $b>0$ atque adeo $b>af$, evidens est signo radicali tribui debere signum —, ut fiat

$$\varDelta=\frac{1}{f}\,l\,\frac{\sqrt{(bb-aaff)}}{b-af}=\frac{1}{2f}\,l\,\frac{b+af}{b-af},$$

et iam habebimus istam summationem

$$\frac{2af}{l\frac{b+af}{b-af}}=b-\cfrac{aaff}{3b-\cfrac{4aaff}{5b-\cfrac{9aaff}{7b-\cfrac{16aaff}{9b-\text{etc.}}}}}$$

unde, cum sit $\frac{b+af}{b-af}>1$, patet valorem huius expressionis fore positivum.

14. Sin autem fuerit b numerus negativus sive si loco b scribatur $-b$, etiamnunc esse debet $b>af$; tum autem erit

$$\varDelta=\frac{1}{2f}\,l\,\frac{b-af}{b+af},$$

qui ergo logarithmus erit negativus, sive

$$\varDelta=-\frac{1}{2f}\,l\,\frac{b+af}{b-af},$$

unde obtinebitur sequens aequatio

$$\frac{-2af}{l\frac{b+af}{b-af}}=-b-\cfrac{aaff}{-3b-\cfrac{4aaff}{-5b-\cfrac{9aaff}{-7b-\cfrac{16aaff}{-9b-\text{etc.}}}}}$$

sive mutatis signis

$$\frac{2af}{l\frac{b+af}{b-af}} = b + \cfrac{aaff}{-3b + \cfrac{4aaff}{5b + \cfrac{9aaff}{-7b + \cfrac{16aaff}{9b + \text{etc.}}}}}$$

cuius ergo fractionis continuae summa aequalis est illi, quam in paragrapho praecedente invenimus. Ista autem aequalitas harum duarum expressionum calculum facienti mox fiet manifesta.

15. Eodem modo evolvamus casum, quo $c = -gg$, pro quo supra invenimus

$$\varDelta = \frac{1}{g}\,\text{A sin.}\frac{ag}{\sqrt{(bb+aagg)}},$$

qui valor per cosinum expressus dabit

$$\varDelta = \frac{1}{g}\,\text{A cos.}\frac{b}{\sqrt{(bb+aagg)}},$$

unde patet per tangentem istum valorem adhuc fore simpliciorem; fit scilicet

$$\varDelta = \frac{1}{g}\,\text{A tang.}\frac{ag}{b},$$

quamobrem pro hoc casu prodit ista summatio

$$\frac{ag}{\text{A tang.}\frac{ag}{b}} = b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \cfrac{16aagg}{9b + \text{etc.}}}}}$$

ubi nulla amplius limitatione est opus.

DE FRACTIONIBUS CONTINUIS A LOGARITHMIS PENDENTIBUS

16. Perpendamus nunc etiam aliquos casus speciales in utraque forma contentos, et quoniam iam observavimus binas formas in § 13 et 14 inter se congruere, utamur priori, qua erat

$$\frac{2af}{l\frac{b+af}{b-af}}=b-\cfrac{aaff}{3b-\cfrac{4aaff}{5b-\cfrac{9aaff}{7b-\text{etc.}}}}$$

ac primo consideremus casum, quo $b=af$, quippe quo evadit summa fractionis

$$\frac{2af}{l\frac{b+af}{b-af}}=0=b-\cfrac{bb}{3b-\cfrac{4bb}{5b-\cfrac{9bb}{7b-\text{etc.}}}}$$

quae per reductionem facile mutatur in hanc

$$0=1-\cfrac{1}{3-\cfrac{4}{5-\cfrac{9}{7-\cfrac{16}{9-\text{etc.}}}}}$$

17. In ista igitur forma nihilo aequali necesse est, ut denominator primae fractionis sit $=1$ ideoque

$$1=3-\cfrac{4}{5-\cfrac{9}{7-\text{etc.}}}\qquad\text{sive}\qquad 0=2-\cfrac{4}{5-\cfrac{9}{7-\text{etc.}}}$$

Hic igitur ob eandem rationem necesse est, ut prior denominator fiat $=2$, ita ut

$$2=5-\cfrac{9}{7-\cfrac{16}{9-\text{etc.}}}\qquad\text{sive}\qquad 0=3-\cfrac{9}{7-\cfrac{16}{9-\text{etc.}}}$$

Hic iterum primus denominator debet esse $= 3$ ideoque

$$3 = 7 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}} \qquad \text{sive} \qquad 0 = 4 - \cfrac{16}{9 - \cfrac{25}{11 - \text{etc.}}}$$

Denuo igitur primus denominator esse debet $= 4$, ita ut

$$4 = 9 - \cfrac{25}{11 - \text{etc.}},$$

atque hoc modo patet istam relationem eodem ordine in infinitum locum habere, in quo ipso criterium veritatis huius aequationis est situm.

18. Quoniam in hac forma numerus b maior esse debet quam af, statuamus nunc $b = 2af$ et nanciscemur sequentem summationem

$$\frac{2af}{l3} = 2af - \cfrac{aaff}{6af - \cfrac{4aaff}{10af - \cfrac{9aaff}{14af - \text{etc.}}}}$$

quae reducitur ad hanc formam mere numericam

$$\frac{2}{l3} = 2 - \cfrac{1}{6 - \cfrac{4}{10 - \cfrac{9}{14 - \cfrac{16}{18 - \text{etc.}}}}}$$

19. Simili modo omnes litterae ex calculo expelli possunt, si pro b accipiatur multiplum ipsius af. Sit enim in genere $b = naf$ ac prodit

$$\frac{2af}{l\frac{n+1}{n-1}} = naf - \cfrac{aaff}{3naf - \cfrac{4aaff}{5naf - \cfrac{9aaff}{7naf - \text{etc.}}}}$$

quae fractio reducitur ad formam sequentem

$$\frac{2}{l\frac{n+1}{n-1}} = n - \cfrac{1}{3n - \cfrac{4}{5n - \cfrac{9}{7n - \text{etc.}}}}$$

unde intelligitur, quemadmodum omnes logarithmos per fractiones continuas exprimi conveniat.

20. Possent hic pro n numeri fracti accipi, tum autem priores termini in singulis membris prodirent fracti, quas quidem per reductionem ad integros revocare liceret; verum huiusmodi casus facillime ex forma generali derivari possunt scribendo statim $b = n$ et $af = m$; tum enim habebimus

$$\frac{2m}{l\frac{n+m}{n-m}} = n - \cfrac{mm}{3n - \cfrac{4mm}{5n - \cfrac{9mm}{7n - \text{etc.}}}}$$

unde, si loco m scribatur $\sqrt{k}$, erit

$$\frac{2\sqrt{k}}{l\frac{n+\sqrt{k}}{n-\sqrt{k}}} = n - \cfrac{k}{3n - \cfrac{4k}{5n - \cfrac{9k}{7n - \text{etc.}}}}$$

21. Hinc igitur omnium numerorum integrorum logarithmos hyperbolicos per fractiones continuas exprimere poterimus. Propositus igitur sit in genere numerus integer i ac statuatur $\frac{n+m}{n-m} = i$; erit $\frac{n}{m} = \frac{i+1}{i-1}$. Capiatur ergo $n = i+1$ et $m = i-1$ atque habebimus

$$\frac{2(i-1)}{li} = i + 1 - \cfrac{(i-1)^2}{3(i+1) - \cfrac{4(i-1)^2}{5(i+1) - \cfrac{9(i-1)^2}{7(i+1) - \cfrac{16(i-1)^2}{9(i+1) - \text{etc.}}}}}$$

unde colligimus

$$li = \cfrac{2(i-1)}{i+1-\cfrac{(i-1)^2}{3(i+1)-\cfrac{4(i-1)^2}{5(i+1)-\cfrac{9(i-1)^2}{7(i+1)-\text{etc.}}}}}$$

22. Si huiusmodi fractiones desideremus pro logarithmis numerorum fractorum, statuamus $\frac{n+m}{n-m}=\frac{p}{q}$, unde fit $n=p+q$ et $m=p-q$, quamobrem habebimus

$$l\frac{p}{q} = \cfrac{2(p-q)}{1(p+q)-\cfrac{1(p-q)^2}{3(p+q)-\cfrac{4(p-q)^2}{5(p+q)-\cfrac{9(p-q)^2}{7(p+q)-\text{etc.}}}}}$$

quae forma eo magis est notatu digna, quod satis commode adhiberi potest ad logarithmos proxime investigandos. Eo magis autem istae fractiones continuae convergent, quo minor fuerit fractio $\frac{p-q}{p+q}$.

23. Quo hoc exemplo illustremus, sumamus $p=2$ et $q=1$, unde quidem non adeo vehemens convergentia est expectanda, eritque

$$l2 = \cfrac{2}{3-\cfrac{1}{9-\cfrac{4}{15-\cfrac{9}{21-\text{etc.}}}}}$$

unde sumendo tantum primum membrum $\frac{2}{3}$ in fractione decimali prodit 0,666666, dum ex tabulis habetur $l2=0{,}693147$, ubi error iam satis est exiguus. Capiamus iam bina membra priora

$$\cfrac{2}{3-\cfrac{1}{9}} = \frac{9}{13} = 0{,}6923.$$

Sumendo autem tria membra habebimus

$$\cfrac{2}{3-\cfrac{1}{9-\cfrac{4}{15}}} = \cfrac{2}{3-\cfrac{15}{131}} = \frac{262}{378} = 0{,}693121,$$

qui valor a veritate deficit quantitate 0,000026.

Multo promptior autem deprehendetur convergentia, si sumamus $p=3$ et $q=2$, ut habeamus

$$l\frac{3}{2} = \cfrac{2}{5-\cfrac{1}{15-\cfrac{4}{25-\cfrac{9}{35-\text{etc.}}}}}$$

cuius primum membrum dat $\frac{2}{5}=0{,}400000$; revera autem est $l\frac{3}{2}=0{,}405465108$. Sumtis autem duobus membris

$$\cfrac{2}{5-\cfrac{1}{15}}$$

colligitur $l\frac{3}{2}=0{,}40540$, ubi error tantum in quintam figuram irrepit. Sumantur tria membra

$$\cfrac{2}{5-\cfrac{1}{15-\cfrac{4}{25}}} = \cfrac{2}{5-\cfrac{25}{371}} = 0{,}4054645,$$

ubi error demum in septima figura se manifestat.

24. Ob hunc insignem usum, qui se praeter expectationem obtulit, operae pretium erit talem investigationem in genere expedire; atque in hunc finem utamur formula inter litteras m et n supra § 20 data, ubi fit

$$l\frac{n+m}{n-m}=\cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n-\cfrac{9mm}{7n-\cfrac{16mm}{9n-\text{etc.}}}}}}$$

unde, si capiamus tantum primum membrum, fiet propemodum

$$l\frac{n+m}{n-m}=\frac{2m}{n};$$

sumtis autem binis prioribus membris

$$\cfrac{2m}{n-\cfrac{mm}{3n}}$$

erit iam propius

$$l\frac{n+m}{n-m}=\frac{6mn}{3nn-mm},$$

sumtis vero tribus membris erit

$$l\frac{n+m}{n-m}=\cfrac{2m}{n-\cfrac{mm}{3n-\cfrac{4mm}{5n}}}=\frac{30mnn-8m^3}{15n^3-9mmn}.$$

25. Non adeo autem operosum est has fractiones ulterius continuare; fractionibus enim iam inventis praefigamus fractionem $\frac{0}{1}$, ut obtineamus hanc fractionum progressionem

$$\begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \\ \frac{0}{1}, & \frac{2m}{n}, & \frac{6mn}{3nn-mm}, & \frac{30mnn-8m^3}{15n^3-9mmn}, \end{array}$$

cuius tam numeratores quam denominatores ex binis praecedentibus ad similitudinem serierum recurrentium formari possunt. Tertia scilicet ex prima et secunda formatur ope huius scalae relationis $3n$, $-mm$; quarta vero formatur ex binis praecedentibus ope huius scalae relationis $5n$, $-4mm$.

Pro quinta igitur utendum erit hac scala $7n$, $-9mm$, pro sexta hac $9n$, $-16mm$, et ita porro. Hoc igitur modo facile reperitur fractio quinta

$$\mathrm{V} = \frac{210mn^3 - 110m^3n}{105n^4 - 90mmnn + 9m^4},$$

simili modo

$$\mathrm{VI} = \frac{1890mn^4 - 1470m^3nn + 128m^5}{945n^5 - 1050mmn^3 + 225m^4n}$$

etc.

26. Hic autem imprimis notasse iuvabit has fractiones continuo augeri et per incrementa continuo minora ad veritatem accedere. Incrementa autem ista egregio ordine procedunt, uti videre hic licet:

$$\mathrm{II} - \mathrm{I} = \frac{2m}{n},$$

$$\mathrm{III} - \mathrm{II} = \frac{2m^3}{n(3nn - mm)},$$

$$\mathrm{IV} - \mathrm{III} = \frac{2\cdot 4m^5}{(3nn - mm)(15n^3 - 9mmn)},$$

$$\mathrm{V} - \mathrm{IV} = \frac{2\cdot 4\cdot 9m^7}{(15n^3 - 9mmn)(105n^4 - 90mmnn + 9m^4)},$$

$$\mathrm{VI} - \mathrm{V} = \frac{2\cdot 4\cdot 9\cdot 16m^9}{(105n^4 - 90mmnn + 9m^4)(945n^5 - 1050mmn^3 + 225m^4n)},$$

unde patet, quo maior fuerit numerus n prae m, eo citius has differentias tam fieri exiguas, ut sine errore negligi queant.

DE FRACTIONIBUS CONTINUIS AB ARCUBUS CIRCULARIBUS PENDENTIBUS

27. Ex § 15 arcus circuli, cuius tangens est $\frac{ag}{b}$, ita per fractionem exprimetur, ut sit

$$A\,\text{tang.}\frac{ag}{b} = \cfrac{ag}{b + \cfrac{aagg}{3b + \cfrac{4aagg}{5b + \cfrac{9aagg}{7b + \text{etc.}}}}}$$

Ponamus nunc ad similitudinem superiorum formarum $ag = m$ et $b = n$ atque habebimus

$$A\,\text{tang.}\frac{m}{n} = \cfrac{m}{n + \cfrac{mm}{3n + \cfrac{4mm}{5n + \cfrac{9mm}{7n + \text{etc.}}}}}$$

quae forma eo citius convergit, quo maior fuerit numerus n prae m; unde patet etiam hanc expressionem cum fructu ad calculum accommodari posse.

28. Incipiamus a casu, quo $m = 1$ et $n = 1$, quo fit

$$A\,\text{tang.}\frac{m}{n} = \frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \text{etc.}}}}}}$$

quae quidem fractio non adeo convergit; attamen videamus, quomodo paullatim ad veritatem accedat, quandoquidem novimus esse $\frac{\pi}{4} = 0{,}78539816339$. Ac primum quidem membrum dabit

$$\frac{\pi}{4} = \frac{1}{1} \quad \text{(nimis magnum);}$$

duo membra praebent

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3}} = \frac{3}{4} \quad \text{(nimis parvum);}$$

tria membra dant

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{4}{5}}} = \frac{19}{24} = 0{,}7916 \text{ (nimis magnum).}$$

Sumantur quatuor membra, ut fiat

$$\frac{\pi}{4} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{4}{5+\cfrac{9}{7}}}} = \frac{40}{51} = 0{,}7843 \text{ [nimis parvum],}$$

ubi error demum in tertia figura deprehenditur.

Ceterum haec fractio continua similis fere est illi, quam olim BROUNCKERUS[1]) in medium protulit, quae ita se habebat

$$\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

Manifestum autem est nostram fractionem multo magis convergere; neque minus concinna est censenda.

29. Quo autem fractionem continuam magis convergentem nanciscamur, statuamus A tang. $\frac{m}{n} = 30^\circ$; cuius tangens cum sit $\frac{1}{\sqrt{3}}$, ne numerus n fiat irrationalis, sumamus $m = \sqrt{3}$ et $n = 3$; hinc igitur fiet

$$\frac{\pi}{6} = \cfrac{\sqrt{3}}{3+\cfrac{3}{9+\cfrac{12}{15+\cfrac{27}{21+\cfrac{48}{27+\text{etc.}}}}}}$$

1) Vide notam p. 227. A. L.

quae forma reducitur ad sequentem

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3+\cfrac{4}{15+\cfrac{9}{7+\cfrac{16}{27+\cfrac{25}{11+\text{etc.}}}}}}}$$

pro qua evolvenda quaeramus primo proxime valorem $\frac{\pi}{6\sqrt{3}}$, qui est 0,3022998. Nunc vero primum membrum praebet

$$\frac{\pi}{6\sqrt{3}} = 0,3333;$$

duo autem priora praebent

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3}} = \frac{3}{10} = 0,3000;$$

tria membra dant

$$\frac{\pi}{6\sqrt{3}} = \cfrac{1}{3+\cfrac{1}{3+\cfrac{4}{15}}} = \frac{49}{162} = 0,30247,$$

ubi error quartam demum figuram afficit.

30. Multo promptior autem convergentia procurari potest, dum angulum rectum in duas partes secamus, quemadmodum olim[1]) ostendi esse

$$\text{A tang.}\,\frac{1}{2} + \text{A tang.}\,\frac{1}{3} = \text{A tang.}\,1 = \frac{\pi}{4}.$$

Sic igitur duas fractiones continuas reperiemus, quarum summa dabit valorem ipsius $\frac{\pi}{4}$, quae erunt

1) Vide *Introductionem in analysin infinitorum*, Lausannae 1748, t. I cap. VIII, § 142; LEONHARDI EULERI *Opera omnia*, series I, vol. 8. A. L.

$$\text{A tang.}\frac{1}{2} = \cfrac{1}{2+\cfrac{1}{6+\cfrac{4}{10+\cfrac{9}{14+\text{etc.}}}}} \quad \text{et} \quad \text{A tang.}\frac{1}{3} = \cfrac{1}{3+\cfrac{1}{9+\cfrac{4}{15+\cfrac{9}{21+\text{etc.}}}}}$$

Manifestum autem est has ambas fractiones et potissimum posteriorem vehementer convergere.

31. Convertamus vero etiam nostram fractionem continuam generalem in fractiones communes; ac ex primo membro solo reperimus

$$\text{A tang.}\frac{m}{n} = \frac{m}{n};$$

ex duobus membris prodit

$$\text{A tang.}\frac{m}{n} = \frac{3mn}{3nn+mm};$$

tria membra praebent

$$\text{A tang.}\frac{m}{n} = \frac{15mnn+4m^3}{15n^3+9mmn}.$$

Sumantur quatuor membra, unde fit

$$\text{A tang.}\frac{m}{n} = \frac{105mn^3+55m^3n}{105n^4+90mmnn+9m^4}.$$

Quodsi nunc his fractionibus praefigatur ut supra $\frac{0}{1}$, orietur haec progressio

$$\begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \frac{0}{1}, & \frac{m}{n}, & \frac{3mn}{3nn+mm}, & \frac{15mnn+4m^3}{15n^3+9mmn}, & \frac{105mn^3+55m^3n}{105n^4+90mmnn+9m^4}, \end{array}$$

cuius singuli termini itidem ex praecedentibus binis secundum certam legem formari possunt, scilicet

pro III scala relationis est $3n, \; +mm,$

pro IV scala relationis est $5n, \; +4mm,$

pro V scala relationis est $7n, \; +9mm$

etc.

METHODUS FACILIS INVENIENDI INTEGRALE HUIUS FORMULAE

$$\int \frac{\partial x}{x} \cdot \frac{x^{n+p} - 2x^n \cos.\zeta + x^{n-p}}{x^{2n} - 2x^n \cos.\theta + 1}$$

CASU QUO POST INTEGRATIONEM PONITUR VEL $x = 1$ VEL $x = \infty$

Conventui exhibita die 18. Martii 1776

Commentatio 620 indicis Enestroemiani
Nova acta academiae scientiarum Petropolitanae 3 (1785), 1788, p. 3—24
Summarium ibidem p. 161—164

SUMMARIUM

La méthode que l'illustre Euler a mise en usage pour trouver l'intégrale de la formule annoncée dans le titre de ce mémoire, est celle qu'on employe ordinairement dans l'intégration de cette espèce de fractions, savoir de les transformer en fractions partielles, selon les facteurs du dénominateur, et d'intégrer chacune de ces fractions simples séparément. Mais pour peu qu'on considère avec attention la formule en question, quiconque connoît l'esprit de cette méthode, sera peut-être surpris de la voir mise en usage pour l'intégration d'une formule aussi compliquée, et s'attendra ou à des résultats plus compliqués encore, ou, s'il en apperçoît la simplicité, il s'attendra à des artifices de calcul peu communs et capables de répandre de l'intérêt sur un sujet qui en paroît d'abord peu susceptible; et c'est effectivement ce qui fait le prix de ce mémoire.

L'Auteur commence par donner à sa formule la forme

$$\int \frac{\partial x}{x} \cdot \frac{x^p - 2\cos.\zeta + x^{-p}}{x^n - 2\cos.\theta + x^{-n}},$$

qu'elle prend lorsqu'on divise le numérateur et le dénominateur par x^n, où le dénominateur est un produit de n facteurs simples trinomiaux de la forme $x^1 - 2\cos.\omega + x^{-1}$; et la forme de chaque fraction partielle qui naît de la résolution de la fraction proposée, devient

$$\frac{2(\cos. p\omega - \cos. \zeta)}{n \sin. \theta} \cdot \frac{\sin. \omega}{x^1 - 2\cos. \omega + x^{-1}},$$

les valeurs de l'angle ω étant au nombre de n, savoir

$$\frac{\theta}{n}, \quad \frac{\theta + 2\pi}{n}, \quad \frac{\theta + 4\pi}{n}, \ldots \quad \frac{\theta + 2(n-1)\pi}{n}.$$

En multipliant donc chacune de ces fractions par $\frac{\partial x}{x}$, et mettant après l'intégration $x = 1$, l'intégrale de la formule proposée devient

$$\frac{Q}{n \sin. \theta} - \frac{R \cos. \zeta}{n \sin. \theta},$$

les valeurs de Q et R étant

$$R = \frac{n\pi - \theta}{n} + \frac{(n-2)\pi - \theta}{n} + \frac{(n-4)\pi - \theta}{n} + \text{etc.},$$

$$Q = \frac{n\pi - \theta}{n} \cos. \frac{p\theta}{n} + \frac{(n-2)\pi - \theta}{n} \cos. \frac{p}{n}(\theta + 2\pi) + \text{etc.};$$

où il est d'abord clair que $R = \pi - \theta$, de sorte que tout revient à déterminer la somme de la série Q.

Pour cet effet l'Auteur considère les deux séries suivantes de n termes

$$t = \cos.(\alpha + 2\beta) + \cos.(\alpha + 4\beta) + \cdots + \cos.(\alpha + 2n\beta),$$
$$u = \cos.(\alpha + 2\beta) + 2\cos.(\alpha + 4\beta) + \cdots + n\cos.(\alpha + 2n\beta),$$

dont il lui est facile de trouver les sommes par des transformations et combinaisons tirées des principes de son calcul des sinus. Ensuite il considère cette progression formée des deux précédentes

$$V = (a + b)\cos.(\alpha + 2\beta) + (a + 2b)\cos.(\alpha + 4\beta) + \cdots + (a + nb)\cos.(\alpha + 2n\beta)$$

de manière que $V = at + bu$, ce qui, les sommes des progressions t et u étant trouvées, lui fournit la somme V. Or en comparant entre elles les progressions V et Q, on peut déterminer les quantités a, b, α, β, et on parvient enfin à cette expression

$$Q = \frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{\sin. \frac{p\pi}{n}},$$

de façon que l'intégrale cherchée sera pour $x = 1$

$$\frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{n \sin. \theta \sin. \frac{\pi p}{n}} - \frac{(\pi - \theta)\cos. \zeta}{n \sin. \theta},$$

et deux fois plus grande, lorsqu'après l'intégration on met $x = \infty$.

Il y a à faire les remarques suivantes par rapport à cette intégration.

1. L'exposant p doit être plus petit que l'exposant n; car autrement la fraction proposée seroit une fraction impropre: elle contiendroit des parties entières dont il faudroit chercher séparément l'intégrale et l'ajouter à l'intégrale trouvée d'après la méthode exposée ici. (Cette opération se trouve détaillée dans le mémoire suivant.)

2. Les opérations par lesquelles on est parvenu à l'intégrale de la formule proposée, ne sauroient avoir lieu, à moins que l'exposant p ne soit un nombre entier. Cependant M. EULER fait voir que les cas où l'exposant p est une fraction quelconque, peuvent toujours être réduits à d'autres où les exposans sont des nombres entiers; et que par conséquent cette circonstance ne change rien aux résultats; mais que, quel que soit le nombre p, entier ou fractionnaire, pourvu que $p < n$, l'intégrale reste comme elle a été assignée.

3. On peut même donner à cet exposant p une valeur imaginaire, pourvu qu'elle soit telle que la formule différentielle reste réelle. Ainsi, en mettant $p = q\sqrt{-1}$, la formule à intégrer devient [pour $\zeta = 90^0$]

$$\int \frac{\partial x}{x} \cdot \frac{2 \cos. q l x}{x^n - 2 \cos. \theta + x^{-n}},$$

dont l'intégrale, prise depuis $x = 0$ jusqu'à $x = 1$, est

$$\frac{\pi}{n \sin. \theta} \cdot \frac{e^{-\frac{q}{n}(\pi-\theta)} - e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}$$

et deux fois plus grande pour les termes d'intégration $x = 0$ et $x = \infty$; vérité de laquelle, comme l'Auteur ajoute, il seroit difficile de donner une démonstration directe.

Ces remarques sont suivies de quelques autres très-propres à répandre du jour tant sur cette intégration que sur bien d'autres, et qui tendent toutes à démontrer que l'intégrale, telle qu'elle a été trouvée, est vraye, quelles que soyent les valeurs de p, n et θ, soit entières, rompues, ou même imaginaires, les seuls cas exceptés où $p - n$ est une quantité positive et réelle.

1. Denotet S integrale huius formulae generaliter sumtum, ita ut quaeri debeat valor ipsius S casu, quo statuitur $x = 1$; ubi primum observo formam propositam multo concinniorem reddi, si fractionis numerator et denominator per x^n dividantur; tum enim habebimus

$$S = \int \frac{\partial x}{x} \cdot \frac{x^p - 2 \cos. \zeta + x^{-p}}{x^n - 2 \cos. \theta + x^{-n}}.$$

Hic statim patet denominatorem $x^n - 2\cos.\theta + x^{-n}$ semper in n factores resolvi posse, qui singuli sint formae $x^1 - 2\cos.\omega + x^{-1}$, ubi angulum ω ita capi oportet, ut, dum iste evanescit, simul etiam ipse denominator ad nihilum redigatur.

2. Posito autem isto factore $x^1 - 2\cos.\omega + x^{-1} = 0$, unde fit

$$x = \cos.\omega + \sqrt{-1} \cdot \sin.\omega,$$

inde in genere colligitur

$$x^\lambda = \cos.\lambda\omega + \sqrt{-1} \cdot \sin.\lambda\omega \quad \text{et} \quad x^{-\lambda} = \cos.\lambda\omega - \sqrt{-1} \cdot \sin.\lambda\omega.$$

Hinc ergo denominator istum accipiet valorem $2\cos.n\omega - 2\cos.\theta$, qui igitur evanescet, si pro $n\omega$ sumatur aliquis ex his valoribus

$$\theta, \quad \theta + 2\pi, \quad \theta + 4\pi, \quad \theta + 6\pi, \quad \theta + 8\pi \quad \text{etc.};$$

quare, cum numerus horum valorum debeat esse $= n$, omnes valores anguli ω erunt sequentes

$$\frac{\theta}{n}, \quad \frac{\theta + 2\pi}{n}, \quad \frac{\theta + 4\pi}{n}, \quad \frac{\theta + 6\pi}{n}, \; \ldots \; \frac{\theta + 2(n-1)\pi}{n}.$$

Praeterea vero cum sit $\cos.n\omega = \cos.\theta$, erit etiam $\sin.n\omega = \sin.\theta$.

3. Cum igitur denominator habeat n factores huius formae

$$x^1 - 2\cos.\omega + x^{-1},$$

nostra fractio, quicunque fuerit eius numerator, in n fractiones simplices resolvi poterit, quarum denominatores erunt illi n factores denominatoris. Scribamus igitur brevitatis gratia Π loco numeratoris $x^p - 2\cos.\zeta + x^{-p}$ atque haec fractio

$$\frac{\Pi}{x^n - 2\cos.\theta + x^{-n}}$$

resolvetur in n fractiones simplices, quarum singulae hanc habebunt formam

$$\frac{P}{x^1 - 2\cos.\omega + x^{-1}};$$

quocirca statuamus

$$\frac{\Pi}{x^n - 2\cos.\theta + x^{-n}} = \frac{P}{x^1 - 2\cos.\omega + x^{-1}} + R,$$

ubi littera R omnes reliquas complectatur fractiones, unde statim habebimus

$$\frac{\Pi(x^1 - 2\cos.\omega + x^{-1})}{x^n - 2\cos.\theta + x^{-n}} = P + R(x^1 - 2\cos.\omega + x^{-1}).$$

Quodsi iam faciamus $x^1 - 2\cos.\omega + x^{-1} = 0$, hinc colligemus numeratorem P; erit enim

$$P = \frac{\Pi(x^1 - 2\cos.\omega + x^{-1})}{x^n - 2\cos.\theta + x^{-n}},$$

siquidem in hac aequatione ponatur $x = \cos.\omega + \sqrt{-1}\cdot\sin.\omega$.

4. At vero iam vidimus, si ipsi x hunc valorem tribuamus, illius fractionis tam numeratorem quam denominatorem evanescere, quamobrem secundum regulam notissimam loco numeratoris et denominatoris sua scribamus differentialia ac prodibit

$$P = \frac{\Pi(x^1 - x^{-1})}{nx^n - nx^{-n}}.$$

Nunc igitur si loco x valor assignatus scribatur, primo pro Π nanciscemur hunc valorem

$$\Pi = 2\cos.p\omega - 2\cos.\zeta;$$

ex fractione autem oritur iste valor $\frac{\sin.\omega}{n\sin.n\omega}$; quae ergo expressio cum sit realis, numerator quaesitus erit

$$P = \frac{2\sin.\omega(\cos.p\omega - \cos.\zeta)}{n\sin.n\omega}.$$

Iam autem vidimus esse $\sin.n\omega = \sin.\theta$, unde iste numerator erit

$$P = \frac{2\sin.\omega(\cos.p\omega - \cos.\zeta)}{n\sin.\theta}.$$

5. Quaelibet igitur fractio partialis ex resolutione fractionis propositae oriunda erit huiusmodi

$$\frac{2(\cos.p\omega - \cos.\zeta)}{n\sin.\theta}\cdot\frac{\sin.\omega}{x^1 - 2\cos.\omega + x^{-1}};$$

in qua forma si angulo ω successive tribuantur omnes valores supra assignati, qui erant

$$\frac{\theta}{n}, \quad \frac{\theta + 2\pi}{n}, \quad \frac{\theta + 4\pi}{n}, \quad \ldots \quad \frac{\theta + 2(n-1)\pi}{n},$$

orientur omnes fractiones partiales, quae in unam summam collectae ipsam formam propositam $\frac{x^p - 2\cos. \zeta + x^{-p}}{x^n - 2\cos. \theta + x^{-n}}$ producere debebunt, unde etiam singulae in $\frac{\partial x}{x}$ ductae et integratae, tum vero in unam summam collectae, exhibebunt integrale quaesitum S.

6. Consideremus igitur fractionem

$$\frac{\sin. \omega}{x^1 - 2\cos. \omega + x^{-1}},$$

quae ducta in $\frac{\partial x}{x}$ praebet

$$\frac{\partial x \sin. \omega}{x^2 - 2x\cos. \omega + 1},$$

cuius integrale ita sumtum, ut evanescat posito $x = 0$, constat esse

$$= \text{A tang.} \frac{x \sin. \omega}{1 - x\cos. \omega}.$$

Hinc igitur ex hac fractione partiali oritur ista pars integralis

$$\frac{2(\cos. p\omega - \cos. \zeta)}{n \sin. \theta} \cdot \text{A tang.} \frac{x \sin. \omega}{1 - x\cos. \omega},$$

unde ergo facile deducuntur omnes n partes integralis quaesiti, si loco ω ordine omnes eius valores assignati substituantur atque in unam summam colligantur.

7. Quoniam autem hoc loco eum tantum integralis valorem postulamus, qui oritur posito $x = 1$, hoc casu fiet

$$\text{A tang.} \frac{x \sin. \omega}{1 - x\cos. \omega} = \text{A tang.} \frac{\sin. \omega}{1 - \cos. \omega}.$$

At vero ista formula $\frac{\sin.\omega}{1-\cos.\omega}$ exprimit cotangentem anguli $\frac{1}{2}\omega$ ideoque tangentem anguli $\frac{\pi-\omega}{2}$, ita ut hoc casu pars integralis futura sit

$$\frac{\cos.p\omega-\cos.\zeta}{n\sin.\theta}(\pi-\omega).^{1)}$$

Hic autem in transitu notasse iuvabit, si desideretur integrale pro casu $x=\infty$, tum proditurum esse

$$\text{A tang.} -\frac{\sin.\omega}{\cos.\omega};$$

quia igitur est $-\frac{\sin.\omega}{\cos.\omega}$ tangens anguli $\pi-\omega$, cum ante habuissemus $\frac{\pi-\omega}{2}$, hinc patet casu $x=\infty$ etiam totum integrale duplo maius fore quam casu $x=1$.

8. Tribuamus igitur angulo ω successive omnes eius valores eosque ordine hic sistamus eritque

$$\begin{aligned}S=&\frac{\cos.\frac{p\theta}{n}-\cos.\zeta}{n\sin.\theta}\cdot\frac{n\pi-\theta}{n}\\
&+\frac{\cos.\frac{p}{n}(\theta+2\pi)-\cos.\zeta}{n\sin.\theta}\cdot\frac{(n-2)\pi-\theta}{n}\\
&+\frac{\cos.\frac{p}{n}(\theta+4\pi)-\cos.\zeta}{n\sin.\theta}\cdot\frac{(n-4)\pi-\theta}{n}\\
&+\frac{\cos.\frac{p}{n}(\theta+6\pi)-\cos.\zeta}{n\sin.\theta}\cdot\frac{(n-6)\pi-\theta}{n}\\
&+\text{etc.},\end{aligned}$$

quarum formularum numerus debet esse $=n$. Haec autem expressio statim in duas partes distinguatur hoc modo indicandas

$$S=\frac{Q}{n\sin.\theta}-\frac{R\cos.\zeta}{n\sin.\theta},$$

1) Quae formula non valet, nisi sit $0<\omega<2\pi$, unde sequitur haec conditio

$$0<\theta<2\pi.$$

A. L.

ita ut sit

$$Q = \frac{n\pi - \theta}{n} \cos. \frac{p\theta}{n} + \frac{(n-2)\pi - \theta}{n} \cos. \frac{p}{n}(\theta + 2\pi)$$
$$+ \frac{(n-4)\pi - \theta}{n} \cos. \frac{p}{n}(\theta + 4\pi) + \text{etc.},$$

$$R = \frac{n\pi - \theta}{n} + \frac{(n-2)\pi - \theta}{n} + \frac{(n-4)\pi - \theta}{n} + \frac{(n-6)\pi - \theta}{n} + \text{etc.},$$

ita ut iam nobis incumbat in valores litterarum Q et R inquirere.

9. Primo autem statim patet valorem ipsius R esse progressionem arithmeticam decrescentem differentia $\frac{2\pi}{n}$, unde summa n terminorum erit $= \pi - \theta$, ita ut sit $R = \pi - \theta$. At vero inventio progressionis Q maiorem requirit apparatum, quem in finem sequentes investigationes generaliores praemittamus.

10. Consideretur primo progressio ista cosinuum, quorum anguli in progressione arithmetica progrediantur et quorum numerus sit n,

$$t = \cos.(\alpha + 2\beta) + \cos.(\alpha + 4\beta) + \cos.(\alpha + 6\beta) + \cdots + \cos.(\alpha + 2n\beta).$$

Iam multiplicemus utrinque per $2 \sin.\beta$, et cum sit

$$2 \sin.\beta \cos.\gamma = \sin.(\beta + \gamma) - \sin.(\gamma - \beta),$$

proveniet sequens forma

$$2t \sin.\beta = -\sin.(\alpha + \beta) + \sin.(\alpha + 3\beta) + \sin.(\alpha + 5\beta) + \cdots + \sin.(\alpha + (2n+1)\beta),$$
$$- \sin.(\alpha + 3\beta) - \sin.(\alpha + 5\beta) - \cdots$$

ubi omnes termini intermedii manifesto se destruunt, ita ut soli extremi remaneant, hincque ergo fiet

$$t = \frac{\sin.(\alpha + (2n+1)\beta) - \sin.(\alpha + \beta)}{2 \sin.\beta}.$$

11. Deinde vero iidem cosinus combinentur cum numeris naturalibus 1, 2, 3, . . . n ac statuatur

$$u = 1 \cos.(\alpha + 2\beta) + 2 \cos.(\alpha + 4\beta) + 3 \cos.(\alpha + 6\beta) + \cdots + n \cos.(\alpha + 2n\beta),$$

qua expressione ducta in $2\sin.\beta$ adhibita resolutione, qua modo sumus usi, consequemur

$$2u\sin.\beta = -\sin.(\alpha+\beta)+\sin.(\alpha+3\beta)+2\sin.(\alpha+5\beta)+\cdots+n\sin.(\alpha+(2n+1)\beta),$$
$$-2\sin.(\alpha+3\beta)-3\sin.(\alpha+5\beta)-\cdots$$

quae forma reducitur ad istam

$$n\sin.(\alpha+(2n+1)\beta)-2u\sin.\beta$$
$$=\sin.(\alpha+\beta)+\sin.(\alpha+3\beta)+\cdots+\sin.(\alpha+(2n-1)\beta),$$

quae vocetur $=v$.

12. Nunc ista series denuo ducatur in $2\sin.\beta$, et cum in genere sit

$$2\sin.\beta\sin.\gamma=\cos.(\gamma-\beta)-\cos.(\gamma+\beta),$$

nanciscemur

$$2v\sin.\beta=\cos.\alpha-\cos.(\alpha+2\beta)-\cos.(\alpha+4\beta)-\cdots-\cos.(\alpha+2n\beta),$$
$$+\cos.(\alpha+2\beta)+\cos.(\alpha+4\beta)+\cdots$$

unde ob terminos medios omnes se destruentes colligitur

$$v=\frac{\cos.\alpha-\cos.(\alpha+2n\beta)}{2\sin.\beta};$$

quare cum sit

$$u=\frac{n\sin.(\alpha+(2n+1)\beta)-v}{2\sin.\beta},$$

hinc obtinemus

$$u=\frac{n\sin.(\alpha+(2n+1)\beta)}{2\sin.\beta}-\frac{\cos.\alpha-\cos.(\alpha+2n\beta)}{4(\sin.\beta)^2}.$$

13. Combinemus nunc ambas summationes modo traditas in genere ac statuamus

$$V=(a+b)\cos.(\alpha+2\beta)+(a+2b)\cos.(\alpha+4\beta)+(a+3b)\cos.(\alpha+6\beta)$$
$$+\cdots+(a+nb)\cos.(\alpha+2n\beta)$$

atque evidens est fore $V = at + bu$, unde loco t et u valoribus substitutis erit

$$V = \frac{a \sin.(\alpha + (2n+1)\beta) - a \sin.(\alpha + \beta)}{2 \sin.\beta} + \frac{bn \sin.(\alpha + (2n+1)\beta)}{2 \sin.\beta}$$
$$- \frac{b \cos.\alpha - b \cos.(\alpha + 2n\beta)}{4(\sin.\beta)^2}.$$

14. Iam satis perspicuum est progressionem, quam supra littera Q exhibuimus, in ista forma generali pro V inventa contineri, quandoquidem utrinque idem terminorum numerus n occurrit atque coefficientes cosinuum seriei Q etiam progressionem arithmeticam constituunt. Quamobrem pro coefficientibus primo faciamus

$$a + b = \frac{n\pi - \theta}{n} \quad \text{et} \quad a + 2b = \frac{(n-2)\pi - \theta}{n},$$

unde deducimus

$$b = -\frac{2\pi}{n} \quad \text{et} \quad a = \frac{(n+2)\pi - \theta}{n}.$$

Nunc etiam angulos inter se coaequemus faciamusque

$$\alpha + 2\beta = \frac{p\theta}{n} \quad \text{et} \quad \alpha + 4\beta = \frac{p}{n}(\theta + 2\pi),$$

unde colligimus

$$\beta = \frac{\pi p}{n} \quad \text{hincque porro} \quad \alpha = \frac{p}{n}(\theta - 2\pi) = -\frac{p}{n}(2\pi - \theta),$$

hocque modo fiet

$$V = Q.$$

At anguli in expressione ipsius V occurrentes erunt primo

$$\alpha + (2n+1)\beta = -\frac{p}{n}(\pi - \theta) + 2\pi p,$$

ubi, cum p sit numerus integer, postrema pars $2\pi p$ totam circumferentiam exprimens omissa est, ex quo habebimus

$$\sin.(\alpha + (2n+1)\beta) = -\sin.\frac{p}{n}(\pi - \theta).$$

Deinde occurrit angulus

$$\alpha+\beta=-\frac{p}{n}(\pi-\theta),$$

cuius sinus est

$$\sin.(\alpha+\beta)=-\sin.\frac{p}{n}(\pi-\theta).$$

Denique erit

$$\alpha+2n\beta=-\frac{p}{n}(2\pi-\theta)+2\pi p$$

et

$$\cos.(\alpha+2n\beta)=\cos.\frac{p}{n}(2\pi-\theta).$$

His igitur valoribus substitutis prodibit

$$Q=V=-\frac{\frac{(n+2)\pi-\theta}{n}\sin.\frac{p}{n}(\pi-\theta)-\frac{(n+2)\pi-\theta}{n}\sin.\frac{p}{n}(\pi-\theta)}{2\sin.\frac{\pi p}{n}}$$

$$+\frac{2\pi\sin.\frac{p}{n}(\pi-\theta)}{2\sin.\frac{\pi p}{n}}+\frac{\frac{2\pi}{n}\cos.\frac{p}{n}(2\pi-\theta)-\frac{2\pi}{n}\cos.\frac{p}{n}(2\pi-\theta)}{4\left(\sin.\frac{\pi p}{n}\right)^2},$$

quae expressio manifesto reducitur ad hanc

$$Q=V=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{\sin.\frac{\pi p}{n}}.$$

15. Inventis igitur valoribus litterarum Q et R valor integralis, quem quaerimus, pro casu $x=1$ erit

$$S=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{\pi p}{n}}-\frac{(\pi-\theta)\cos.\zeta}{n\sin.\theta}.$$

Sin autem integrale quaeratur a termino $x=0$ usque ad $x=\infty$, eius valor duplo maior evadet.

16. His iam in genere expeditis consideremus casum iam saepius tractatum, quo est $\zeta=90^0$ et $\theta=90^0$ haecque formula integranda proponitur

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+x^{-n}},$$

atque pro eius valore casu $x=1$ habebimus

$$S=\frac{\pi\sin.\frac{p\pi}{2n}}{n\sin.\frac{p\pi}{n}},$$

quae ob $\sin.\frac{p\pi}{n}=2\sin.\frac{p\pi}{2n}\cos.\frac{p\pi}{2n}$ abit in formulam illam notissimam

$$\frac{\pi}{2n\cos.\frac{p\pi}{2n}}=\frac{\pi}{2n}\sec.\frac{p\pi}{2n}.$$

Sin autem tantum sumamus $\zeta=90^0$, ut formula integranda sit

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}},$$

eius valor ab $x=0$ usque ad $x=1$ extensus erit

$$\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{p\pi}{n}},$$

quae expressio reducitur ad hanc

$$\frac{\pi}{n\sin.\theta}\left(\cos.\frac{p\theta}{n}-\sin.\frac{p\theta}{n}\cot.\frac{p\pi}{n}\right).$$

OBSERVATIONES IN INTEGRATIONEM TRADITAM

I. Primum hic observo terminum medium in numeratore exhibitum nullo modo integrationem turbare, quoniam, si solus adesset, integratio nulla laboraret difficultate; tum enim formula

$$\int\frac{\partial x}{x}\cdot\frac{1}{x^n-2\cos.\theta+x^{-n}}$$

reducitur ad hanc formam

$$\int\frac{x^{n-1}\partial x}{x^{2n}-2x^n\cos.\theta+1},$$

quae posito $x^n = y$ abit in hanc

$$\frac{1}{n}\int\frac{\partial y}{y^2-2y\cos.\theta+1},$$

cuius integrale est

$$\frac{1}{n\sin.\theta}\text{ A tang.}\frac{y\sin.\theta}{1-y\cos.\theta},$$

cuius valor casu $x=1$ fit

$$\frac{1}{n\sin.\theta}\text{ A tang.}\frac{\sin.\theta}{1-\cos.\theta}=\frac{\pi-\theta}{2n\sin.\theta},$$

qui ductus in $-2\cos.\zeta$ praebet illam ipsam partem hinc oriundam in forma supra inventa, quamobrem superfluum foret hunc terminum in calculo retinere; unde hanc formam integralem sumus contemplaturi

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}},$$

cuius valorem casu $x=1$ invenimus

$$=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{p\pi}{n}},$$

quem brevitatis gratia littera P designemus, ita ut sit

$$P=\frac{\pi\sin.\frac{p}{n}(\pi-\theta)}{n\sin.\theta\sin.\frac{p\pi}{n}}.$$

Tum vero etiam invenimus casu $x=\infty$ valorem huius formulae esse $=2P$.

II. Secundo loco probe notari oportet exponentem p necessario minorem esse debere quam exponentem n, quia alioquin fractio foret spuria et variabilis x in numeratore tot vel plures dimensiones esset habitura quam in denominatore. Quoties autem hoc evenit, integrali praeter partes, quas per resolutionem in fractiones partiales sumus nacti, quantitas quaedam integra adiici debet, id quod in nostra solutione non est factum, quamobrem tales casus hinc prorsus excludi convenit. Ceterum quolibet casu has partes integras facile erit adiicere ad partes, quas nobis nostra methodus suppeditabit.

III. Ex ipsa solutione, quam dedimus, perspicuum est exponentem p necessario integrum statui debere, quia alias operationes ibi exhibitae locum habere non possent; unde eo magis mirum videbitur, quod conclusiones inventae subsistere queant, etiamsi iste exponens p fuerit numerus fractus quicunque, dummodo minor quam n, propterea quod hos casus semper ad exponentes integros reducere licet. Ad hoc ostendendum ponamus esse $p = \frac{q}{\lambda}$ atque forma nostra posito $x = z^\lambda$ reducetur ad hanc formam

$$\lambda \int \frac{\partial z}{z} \cdot \frac{z^q + z^{-q}}{z^{\lambda n} - 2\cos.\theta + z^{-\lambda n}};$$

ubi cum omnes exponentes sint integri ac pro terminis integrationis, qui erant $x = 0$, $x = 1$ et $x = \infty$, etiam fiat $z = 0$, $z = 1$ et $z = \infty$, pro $z = 1$ valor integralis erit

$$\frac{\lambda\pi \sin.\frac{q}{\lambda n}(\pi - \theta)}{\lambda n \sin.\theta \sin.\frac{q\pi}{\lambda n}},$$

qui restituto loco $\frac{q}{\lambda}$ valore p abit in hunc

$$\frac{\pi \sin.\frac{p}{n}(\pi - \theta)}{n \sin.\theta \sin.\frac{p\pi}{n}},$$

quae expressio cum superiore prorsus congruit. Atque hinc intelligitur, etiamsi[1]) exponenti p valores irrationales tribuantur, dumne superent exponentem n, semper hoc evenire debere.

IV. Hic iam quaestio oritur maximi momenti, utrum etiam exponenti p dare liceat valores imaginarios necne. Hoc autem affirmandum videtur, quandoquidem imaginaria certe non sint maiora quam n; unde concludimus, dummodo valor ipsius p ita capiatur imaginarius, ut ipsa formula differentialis maneat realis, tum etiam conclusiones nostras veritati consentaneas esse mansuras.[2]) Hoc autem evenit, si statuamus $p = q\sqrt{-1}$; tum enim, cum in genere sit

$$e^{\varphi\sqrt{-1}} + e^{-\varphi\sqrt{-1}} = 2\cos.\varphi,$$

1) Editio princeps: ... *intelligitur, quominus etiam exponenti* ... Correxit A. L.

2) Quamquam hoc non satis confirmatum est, tamen hic revera ponere licet $p = q\sqrt{-1}$, quia integrale valore absoluto ipsius p manente satis parvo in seriem integram secundum ipsius p potestates progredientem resolvi potest. A. L.

quia nostro casu est $\varphi = qlx$, ipsa formula integralis erit

$$\int \frac{\partial x}{x} \cdot \frac{2 \cos. qlx}{x^n - 2\cos.\theta + x^{-n}}.$$

Nunc igitur videamus, quamnam formam nostrum integrale casu $x=1$ sit recepturum, et quoniam sinus angulorum imaginariorum sunt etiam imaginarii, quandoquidem

$$e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}} = \frac{2}{\sqrt{-1}} \sin.\varphi,$$

loco φ scribamus $\psi\sqrt{-1}$ eritque

$$\sin.\psi\sqrt{-1} = \frac{\sqrt{-1}}{2}(e^{-\psi} - e^{+\psi}),$$

unde in forma integrali erit $\frac{p}{n} = \frac{q\sqrt{-1}}{n}$, ideoque loco ψ scribamus $\frac{q}{n}(\pi - \theta)$ pro numeratore, at $\frac{q\pi}{n}$ pro denominatore, ex quo valor integralis ab $x=0$ ad $x=1$ extensus erit

$$\frac{\pi}{n\sin.\theta} \cdot \frac{e^{-\frac{q}{n}(\pi-\theta)} - e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}.$$

Hinc igitur formemus sequens theorema notatu dignissimum.

THEOREMA

Quodsi ista formula integralis

$$\int \frac{\partial x}{x} \cdot \frac{\cos. qlx}{x^n - 2\cos.\theta + x^{-n}}$$

a termino $x=0$ usque ad $x=1$ extendatur, eius valor semper erit

$$= \frac{\pi}{2n\sin.\theta} \cdot \frac{e^{-\frac{q}{n}(\pi-\theta)} - e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}.$$

Sin autem integrale extendatur ab $x=0$ ad $x=\infty$, valor prodibit duplo maior.

Hoc theorema utique eo maiorem attentionem meretur, quod nulla via patet eius veritatem directe demonstrandi.

V. Revertamur autem ad formam integralem primo expositam, et quoniam numerator duabus constat partibus x^p et x^{-p}, unde summa integralium pro $x = 1$ inventa est $= P$, at pro casu $x = \infty$ duplo maior $= 2P$, hic maxime notatu dignum occurrit, quod pro termino $x = \infty$ utraque pars numeratoris eundem producat valorem $= P$. Semper enim erit integrale ab $x = 0$ ad $x = \infty$ extendendo

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n - 2 \cos. \theta + x^{-n}} = \int \frac{\partial x}{x} \cdot \frac{x^{-p}}{x^n - 2 \cos. \theta + x^{-n}} = P.$$

Ad hoc ostendendum ponamus pro posteriore formula $x = \frac{1}{z}$ eaque induet hanc formam

$$-\int \frac{\partial z}{z} \cdot \frac{z^{+p}}{z^{-n} - 2 \cos. \theta + z^n},$$

quae cum sit priori formae prorsus similis solo signo — excepto, eius valor a termino $z = 0$ usque ad $z = \infty$ negative sumtus primae formulae erit aequalis. Cum autem sit $z = \frac{1}{x}$, isti termini integralis erunt ab $x = \infty$ usque ad $x = 0$; qui ergo si invertantur, etiam signum integralis erit mutandum sicque ipsi priori formulae aequale evadet; quare cum ambae formulae coniunctae summam habeant $= 2P$, utriusque seorsim sumtae valor erit $= P$, unde deducitur sequens theorema notatu pariter dignissimum.

THEOREMA

Istius formulae integralis

$$\int \frac{\partial x}{x} \cdot \frac{x^{\pm p}}{x^n - 2 \cos. \theta + x^{-n}}$$

valor a termino $x = 0$ usque ad $x = \infty$ extensus semper est

$$= P = \frac{\pi \sin. \frac{p}{n}(\pi - \theta)}{n \sin. \theta \sin. \frac{p\pi}{n}}.$$

Evidens autem est hanc aequalitatem pro casu $x = 1$ neutiquam locum habere posse.

VI. Quoniam in nostra formula differentiali tantum occurrit terminus $2 \cos. \theta$, cuius valor idem manet, etiamsi pro θ sumeremus $\theta \pm 2i\pi$, maxime

hic mirum videri debet, quod tum valor integralis maxime diversus sit proditurus, scilicet

$$= \frac{\pi \sin. \frac{p}{n}(\pi - \theta \pm 2i\pi)}{n \sin. \theta \sin. \frac{p\pi}{n}},$$

unde merito quaeritur, quisnam horum valorum veritati sit conformis; ad quod certe nihil aliud responderi potest, nisi quod omnes veritati aeque consentanei sint censendi[1]), id quod eo minus mirum videri debet, quod omnes huiusmodi formulae integrales revera sunt functiones multiformes atque adeo infinitiformes, id quod ex hoc exemplo simplicissimo $\int \frac{\partial x}{1+xx}$ intelligi potest. Cum enim eius integrale exhibeat arcum circuli, cuius tangens est x, tamen autem arcus innumerabiles dentur, quorum eadem sit tangens $= x$, necesse est, ut omnes aeque in hac forma integrali contineantur. Quin etiam in nostro valore invento P loco π quoque scribere licet $\pi + 2i\pi$ eiusque valor nihilominus cum veritate consistere poterit. Verum in huiusmodi integrationibus perpetuo valores minimi desiderari solent hocque modo omnis difficultas e medio est sublata.

VII. Deinde in analysi supra adhibita supposuimus omnes factores denominatoris inter se esse inaequales, id quod utique semper evenit, nisi sit $\cos. \theta = \pm 1$, quippe quibus casibus denominator quadratum involvit; fit enim is

$$= x^{-n}(x^n \pm 1)^2,$$

ex quo patet omnes factores $x^n \pm 1$ bis occurerre debere. Hoc incommodum etiam innuitur per ipsam nostram formulam P, quae casu $\theta = 0$ valorem indicat infinitum. Verum posito $\theta = \pi$ singulare phaenomenon se offert, dum formulae pro P inventae tam numerator quam denominator evanescunt atque adeo fractio determinatum nanciscitur valorem. Ponamus enim $\theta = \pi - \omega$ existente ω infinite parvo eritque $\sin. \theta = \sin. \omega = \omega$; at ob $\pi - \theta = \omega$ in numeratore habebimus $\sin. \frac{p\omega}{n} = \frac{p\omega}{n}$, unde valor ipsius P resultat $\frac{\pi p}{nn \sin. \frac{p\pi}{n}}$; qui cum penitus sit determinatus, nullum plane dubium superesse potest, quin cum veritate conspiret, unde sequens enascitur theorema maxime memorabile.

1) Formula inventa nonnisi hac conditione $0 < \theta < 2\pi$ valet (vide notam p. 271). A. L.

THEOREMA

Proposita formula differentiali

$$\frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2 + x^{-n}} = \frac{x^{n-1}\partial x(x^p + x^{-p})}{(x^n + 1)^2}$$

si eius integrale a termino $x = 0$ *usque ad* $x = 1$ *extendatur, eius valor semper erit*

$$\frac{\pi p}{nn \sin. \frac{p\pi}{n}};$$

sin autem usque ad terminum $x = \infty$ *extendatur, eius valor erit duplo maior*

$$= \frac{2\pi p}{nn \sin. \frac{p\pi}{n}}.$$

DEMONSTRATIO HUIUS THEOREMATIS DIRECTA

Formula ista integralis resolvatur sequenti modo

$$\int \frac{\partial x}{x} \cdot \frac{x^{n+p} + x^{n-p}}{(1 + x^n)^2} = \frac{Q}{1 + x^n} + \int \frac{\partial x}{x} \cdot \frac{R}{1 + x^n}.$$

Sumantur igitur differentialia simulque ducantur in $\frac{x}{\partial x}$ positoque $\partial Q = Q'\partial x$ orietur ista aequatio

$$\frac{x^{n+p} + x^{n-p}}{(1 + x^n)^2} = \frac{Q'x}{1 + x^n} - \frac{nQx^n}{(1 + x^n)^2} + \frac{R}{1 + x^n},$$

quae per $1 + x^n$ multiplicata hoc modo repraesentetur

$$\frac{x^{n+p} + x^{n-p} + nQx^n}{1 + x^n} = Q'x + R,$$

ubi iam Q ita accipi debet, ut illa fractio ad integrum revocetur. Facile autem patet hoc fieri statuendo

$$nQ = -x^p + x^{n-p};$$

tum enim illa fractio fiet

$$\frac{x^{n-p} + x^{2n-p}}{1 + x^n} = x^{n-p},$$

ita ut nunc habeamus

$$x^{n-p} = Q'x + R.$$

Cum igitur sit $Q = \frac{x^{n-p}-x^p}{n}$, erit

$$Q'x = \frac{(n-p)x^{n-p}-px^p}{n}$$

hincque colligitur

$$R = \frac{p}{n}(x^{n-p}+x^p),$$

quocirca formula integralis proposita reducta est ad hanc formam

$$\frac{x^{n-p}-x^p}{n(1+x^n)} + \frac{p}{n}\int\frac{\partial x}{x}\cdot\frac{x^{n-p}+x^p}{1+x^n},$$

quod integrale ita est sumendum, ut evanescat posito $x=0$. Nunc igitur statuamus $x=1$ ac prior pars absoluta evanescit, formulae autem integralis valor per ea, quae dudum[1]) sunt inventa, prodit

$$\frac{p}{n}\cdot\frac{\pi}{n\sin.\frac{p\pi}{n}} = \frac{\pi p}{nn\sin.\frac{p\pi}{n}},$$

qui ergo cum ante invento perfecte congruit.

VIII. Tribuatur nunc etiam in hac postrema forma exponenti p valor imaginarius ponendo $p = q\sqrt{-1}$, et cum, ut ante vidimus, hinc fiat $x^p + x^{-p} = 2\cos.qlx$, formula integralis proposita erit

$$= 2\int\frac{\partial x}{x}\cdot\frac{x^n\cos.qlx}{(1+x^n)^2}.$$

Pro eius valore autem iam ante vidimus fore

$$\sin.\frac{\pi q\sqrt{-1}}{n} = \frac{e^{-\frac{\pi q}{n}} - e^{+\frac{\pi q}{n}}}{2\sqrt{-1}},$$

1) Vide L. EULERI Commentationem 60 (indicis ENESTROEMIANI): *De inventione integralium, si post integrationem variabili quantitati determinatus valor tribuatur*, Miscellanea Berolin. 7, 1743, p. 129; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 58. A. L.

quamobrem valor nostrae formulae ab $x=0$ ad $x=1$ extensus erit

$$\frac{2\pi q}{nn\left(e^{\frac{\pi q}{n}}-e^{-\frac{\pi q}{n}}\right)},$$

unde deducimus sequens theorema omni attentione dignum.

THEOREMA

Si valor istius formulae integralis

$$\int\frac{x^{n-1}\partial x\cos.qlx}{(1+x^n)^2}$$

a termino $x=0$ usque ad $x=1$ extendatur, is semper aequabitur huic formulae

$$\frac{\pi q}{nn\left(e^{\frac{\pi q}{n}}-e^{-\frac{\pi q}{n}}\right)}.$$

Cuius autem theorematis demonstratio ex principiis iam cognitis vix elici posse videtur.

IX. Praeterea etiam perspicuum est methodum, qua usi sumus ad nostram formulam integrandam, subsistere non posse, nisi terminus medius denominatoris binario sit minor, quam ob causam eum hac forma $2\cos.\theta$ expressimus. Quamobrem hinc oritur quaestio maximi momenti, utrum nostrae conclusiones etiamnunc valeant, si terminus ille medius binario maior acciperetur sive si angulus θ foret imaginarius, necne. Verum etiam hoc casu nullum dubium superesse potest, quin formula nostra finalis etiamnunc veritati consentanea sit futura. Ante omnia autem hic est observandum illi termino medio $2\cos.\theta$ valorem negativum tribui convenire, quia alioquin ipse denominator in nihilum abiret, dum quantitas nostra variabilis x a termino 0 usque ad 1 augetur. Hanc ob rem statuamus angulum $\theta=\pi-\eta$ et valor noster integralis erit

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+2\cos.\eta+x^{-n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\sin.\frac{p\eta}{n}}{n\sin.\eta\sin.\frac{\pi p}{n}}.$$

In hac igitur forma faciamus angulum η imaginarium ponendo $\eta=\varphi\sqrt{-1}$

eritque per ea, quae iam supra observavimus, $2\cos.\varphi\sqrt{-1}=e^{\varphi}+e^{-\varphi}$, ita ut iam noster denominator sit

$$x^n+e^{\varphi}+e^{-\varphi}+x^{-n}=\frac{1}{x^n}(x^n+e^{\varphi})(x^n+e^{-\varphi}),$$

quem idcirco statim in duos factores reales formae $x+k$ resolvere licet; tum vero fiet

$$\sin.\eta=\sin.\varphi\sqrt{-1}=\frac{e^{-\varphi}-e^{+\varphi}}{2\sqrt{-1}}$$

similique modo erit

$$\sin.\frac{p}{n}\eta=\sin.\frac{p}{n}\varphi\sqrt{-1}=\frac{e^{-\frac{p\varphi}{n}}-e^{+\frac{p\varphi}{n}}}{2\sqrt{-1}},$$

unde formula nostra integralis emergit realis

$$=\frac{\pi\left(e^{-\frac{p\varphi}{n}}-e^{+\frac{p\varphi}{n}}\right)}{n(e^{-\varphi}-e^{+\varphi})\sin.\frac{\pi p}{n}}.$$

Statuamus autem hic brevitatis gratia $e^{\varphi}=f$, ut sit $e^{-\varphi}=\frac{1}{f}$, atque nostra formula integralis sequentem induet formam

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+\left(f+\frac{1}{f}\right)+x^{-n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\left(f^{\frac{p}{n}}-f^{-\frac{p}{n}}\right)}{n(f-f^{-1})\sin.\frac{p\pi}{n}},$$

id quod tamquam theorema omni attentione dignum spectari potest; ubi per se intelligitur valorem eiusdem integralis usque ad $x=\infty$ extensum fore duplo maiorem.

X. Quodsi iam in hac forma etiam exponenti p valorem imaginarium tribuamus, pariter nullo modo dubitari poterit, quin conclusio nostra vera sit mansura. Ponamus igitur $p=q\sqrt{-1}$ eritque ut ante $x^p+x^{-p}=2\cos.qlx$; tum vero erit

$$\sin.\frac{p\pi}{n}=\frac{e^{-\frac{q\pi}{n}}-e^{+\frac{q\pi}{n}}}{2\sqrt{-1}},$$

pro integralis autem numeratore erit

$$f^{\frac{p}{n}} - f^{-\frac{p}{n}} = 2\sqrt{-1} \cdot \sin.\frac{q}{n} lf.$$

His igitur valoribus substitutis sequens nanciscimur

THEOREMA

Valor istius formulae integralis

$$\int \frac{\partial x}{x} \cdot \frac{\cos.qlx}{x^n + \left(f + \frac{1}{f}\right) + x^{-n}}$$

a termino $x = 0$ *usque ad* $x = 1$ *extensus semper aequabitur formulae*

$$\frac{2\pi \sin.\frac{q}{n} lf}{n(f - f^{-1})\left(e^{\frac{q\pi}{n}} - e^{-\frac{q\pi}{n}}\right)}.$$

XI. Deinde iam pridem[1]) observavi omnia huiusmodi integralia satis commode per series infinitas exprimi posse. Cum enim ista fractio

$$\frac{x^p}{x^n - 2\cos.\theta + x^{-n}} = \frac{x^{n+p}}{x^{2n} - 2x^n \cos.\theta + 1}$$

resolvatur in hanc seriem

$$\frac{1}{\sin.\theta}(x^{n+p}\sin.\theta + x^{2n+p}\sin.2\theta + x^{3n+p}\sin.3\theta + \text{etc.}),$$

integrale istud

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n - 2\cos.\theta + x^{-n}}$$

a termino $x = 0$ usque ad $x = 1$ extensum aequabitur huic seriei infinitae

$$\frac{1}{\sin.\theta}\left(\frac{\sin.\theta}{n+p} + \frac{\sin.2\theta}{2n+p} + \frac{\sin.3\theta}{3n+p} + \frac{\sin.4\theta}{4n+p} + \text{etc.}\right).$$

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1) Vide Commentationem 589 huius voluminis, imprimis p. 204. A. L.

Hinc ergo si p negative caperemus, tum formula nostra principalis

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}}$$

ab $x=0$ ad $x=1$ extensa semper aequabitur huic seriei infinitae geminatae

$$\frac{1}{\sin.\theta}\left\{\begin{array}{l}\dfrac{\sin.\theta}{n+p}+\dfrac{\sin.2\theta}{2n+p}+\dfrac{\sin.3\theta}{3n+p}+\dfrac{\sin.4\theta}{4n+p}+\text{etc.}\\ +\dfrac{\sin.\theta}{n-p}+\dfrac{\sin.2\theta}{2n-p}+\dfrac{\sin.3\theta}{3n-p}+\dfrac{\sin.4\theta}{4n-p}+\text{etc.}\end{array}\right\},$$

quae binis homologis coniungendis contrahitur in hanc seriem

$$\frac{2n}{\sin.\theta}\left(\frac{\sin.\theta}{nn-pp}+\frac{2\sin.2\theta}{4nn-pp}+\frac{3\sin.3\theta}{9nn-pp}+\frac{4\sin.4\theta}{16nn-pp}+\text{etc.}\right).$$

XII. Hinc iam manifesto pro casu, quo ponitur $p=q\sqrt{-1}$, ista series infinita exoritur

$$\frac{2n}{\sin.\theta}\left(\frac{\sin.\theta}{nn+qq}+\frac{2\sin.2\theta}{4nn+qq}+\frac{3\sin.3\theta}{9nn+qq}+\frac{4\sin.4\theta}{16nn+qq}+\text{etc.}\right),$$

quae ergo exprimit valorem huius formulae integralis

$$\int\frac{\partial x}{x}\cdot\frac{2\cos.qlx}{x^n-2\cos.\theta+x^{-n}},$$

scilicet ab $x=0$ ad $x=1$ extensae, ita ut istius seriei summa finito modo expressa sit etiam

$$\frac{\pi}{n\sin.\theta}\cdot\frac{e^{-\frac{q}{n}(\pi-\theta)}-e^{+\frac{q}{n}(\pi-\theta)}}{e^{-\frac{q\pi}{n}}-e^{+\frac{q\pi}{n}}}.$$

Quin etiam facile intelligitur hic quoque angulum θ imaginarium accipi posse. Vidimus enim posito $\theta=\varphi\sqrt{-1}$ fore

$$\sin.\theta=\frac{e^{-\varphi}-e^{+\varphi}}{2\sqrt{-1}}$$

hincque in genere

$$\sin.\lambda\theta=\frac{e^{-\lambda\varphi}-e^{+\lambda\varphi}}{2\sqrt{-1}}.$$

Quare si statuamus $e^{\varphi} = f$, erit

$$\frac{\sin. \lambda\theta}{\sin. \theta} = \frac{f^{\lambda} - f^{-\lambda}}{f - \frac{1}{f}},$$

unde series illa satis concinnam formam accipiet.

XII[a].[1]) Denique operationes, quibus in integratione nostrae formulae sumus usi, consistere nequeunt, nisi exponens n fuerit numerus integer. Interim tamen valor integralis, quem invenimus pro casu vel $x = 1$ vel $x = \infty$, veritati conformis deprehenditur, non solum quando pro n numerus fractus quicunque, sed etiam adeo imaginarius accipitur, quorum prius facile ostenditur. Sit enim $n = \frac{m}{\lambda}$ ac ponatur $x = y^{\lambda}$ atque ob $\frac{\partial x}{x} = \frac{\lambda \partial y}{y}$ orietur haec forma integralis exponentibus integris contenta

$$\int \frac{\lambda \partial y}{y} \cdot \frac{y^{\lambda p} + y^{-\lambda p}}{y^m - 2\cos. \theta + y^{-m}},$$

cuius ergo valor casu $x = 1$ debet esse secundum formulam inventam

$$\frac{\lambda\pi}{m} \cdot \frac{\sin. \frac{\lambda p}{m}(\pi - \theta)}{\sin. \theta \sin. \frac{\lambda p \pi}{m}},$$

qui, si iam loco m valor λn restituatur, manifesto abit in ipsam nostram formulam supra [V] inventam

$$\frac{\pi}{n} \cdot \frac{\sin. \frac{p}{n}(\pi - \theta)}{\sin. \theta \sin. \frac{p\pi}{n}}.$$

Hinc autem nulli amplius dubio relinquitur, quin veritas haec subsistat, etiamsi n fuerit numerus imaginarius.[2]) Ponamus igitur $n = m\sqrt{-1}$ et formula integralis reducetur ad hanc formam

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{2\cos. mlx - 2\cos. \theta},$$

1) In editione principe falso numerus XII iteratur. A. L.

2) Manifestum est hanc conclusionem falsam esse, quotiescunque pro n numerus imaginarius formae $m\sqrt{-1}$ accipitur. A. L.

cuius ergo valor casu $x=1$ certe erit

$$\frac{p}{m\sqrt{-1}}\cdot\frac{e^{\frac{p}{m}(\pi-\theta)}-e^{-\frac{p}{m}(\pi-\theta)}}{\sin.\theta\left(e^{\frac{\pi p}{m}}-e^{-\frac{\pi p}{m}}\right)},$$

ubi mirum videbitur istum valorem semper esse imaginarium, licet ipsa formula differentialis, dum variabilis x a termino 0 usque ad terminum $x=1$ augetur, maneat realis, id quod merito maxime videtur paradoxum. Interim tamen non desunt casus, quibus valor integralis formulae differentialis realis manifesto evadit imaginarius, id quod in ista formula simpliciori

$$\int\frac{\partial x}{x\cos.mlx}$$

ostendisse sufficiet, quae utique, dum x a 0 ad 1 augetur, constanter manet realis. Ad hanc ergo formulam integrandam statuamus $lx=-z$, ubi notetur, dum x a 0 usque ad 1 progreditur, tum quantitatem z ab ∞ usque ad 0 decrescere. Nunc igitur formula nostra integralis erit

$$\int\frac{-\partial z}{\cos.mz};$$

cum vero constet esse

$$\int\frac{\partial\varphi}{\sin.\varphi}=l\,\text{tang.}\frac{1}{2}\varphi,$$

sumamus $\varphi=90^0-mz$ eritque $\partial\varphi=-m\partial z$ hincque

$$\int\frac{-m\partial z}{\cos.mz}=+l\,\text{tang.}\left(45^0-\frac{1}{2}mz\right),$$

quod integrale manifesto evanescit pro termino $z=0$; dum autem ab hoc termino quantitas z in infinitum usque augetur, infinities tangens huius anguli fiet negativa eiusque logarithmus propterea imaginarius, unde non amplius mirabimur, quod formulae differentialis realis integrale evadere possit certis casibus imaginarium.

XIII. Hoc igitur modo evictum est formulae nostrae differentialis propositae

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n-2\cos.\theta+x^{-n}}$$

integrale assignatum a termino $x = 0$ usque ad $x = 1$ semper cum veritate consistere, quicunque valores ternis litteris n, p et θ tribuantur sive integri sive fracti sive etiam imaginarii. Interim tamen dantur casus iam initio indicati, quibus isti valores integrales a veritate aberrabunt, quippe quod semper usu venire debet, quoties exponens p maior est exponente n, quam ob causam sedulo excludere debemus omnes casus, quibus formula $p - n$ evadit realis et positiva. His autem exceptis variae formulae, ad quas hic sumus perducti, ita sunt comparatae, ut maxima attentione dignae videantur simulque non contemnenda incrementa scientiae analyticae promittant.

DE SUMMO USU CALCULI IMAGINARIORUM IN ANALYSI

Conventui exhibita die 18. Martii 1776

Commentatio 621 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 3 (1785), 1788, p. 25—46
Summarium ibidem p. 164—167

SUMMARIUM

Les Géomètres de nos jours connoissent suffisamment la grande utilité du Calcul des Imaginaires; ils savent combien il a contribué à l'avancement de l'Analyse et que dans l'intégration des formules différentielles fractionnaires, qui se fait par la résolution en fractions partielles ayant des dénominateurs en partie imaginaires, on ne sauroit se passer de ce calcul. Feu M. EULER avoit fait, à la vérité, quelques tentatives de dégager l'intégration des formules rationnelles de tout emploi des Imaginaires, et quoiqu'il y ait réussi en partie, le succès n'avoit pas été parfaitement heureux dans les cas, où le dénominateur a deux ou plusieurs facteurs égaux. On ne sauroit donc se passer entièrement des Imaginaires et on rencontre parfois des formules intégrales qui paroissent se réfuser à toute voye d'intégration, à moins qu'on n'ait recours aux Imaginaires. C'est ce que l'Auteur se propose de montrer par un nouvel exemple frappant dans le cours de ce mémoire.

Le cas que M. EULER a choisi pour cet effet lui a été fourni par le mémoire précédent. Car ayant trouvé

$$\frac{\pi \sin. \frac{\theta p}{n}}{n \sin. \theta \sin. \frac{\pi p}{n}}$$

pour l'intégrale

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2\cos.\theta + x^{-n}},$$

prise depuis $x=0$ jusqu'à $x=1$, il observe qu'en gardant les mêmes termes d'intégration, on peut déduire de là

$$\int \frac{\partial x}{x\,lx}\cdot\frac{x^p-x^{-p}}{x^n+2\cos.\theta+x^{-n}}=\frac{\pi}{n\sin.\theta}\int\frac{\partial p\,\sin.\frac{\theta p}{n}}{\sin.\frac{\pi p}{n}},$$

p étant regardé comme une quantité variable et l'intégrale prise de manière qu'elle évanouisse en mettant $p=0$. Et c'est l'intégration de cette formule, ou bien, pour éviter les fractions, de celle-ci $\int\frac{\partial\varphi\,\sin.m\varphi}{\sin.n\varphi}$, qui fait le sujet de ce mémoire.

L'Auteur commence par dégager cette expression des quantités angulaires, en mettant $t=\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$ et $u=\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi$, ce qui le conduit à cette expression purement algébrique

$$\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^m-t^{-m}}{t^n-t^{-n}},$$

pour l'intégration de laquelle il faut soigneusement distinguer les cas où l'exposant m surpasse n, des cas où le contraire arrive, vu que dans le premier cas la fraction est impropre et contient des entiers dont il faut avant tout déterminer les intégrales, ce qui étant fait, le reste se réduit à trouver l'intégrale de la formule proposée pour les cas où l'exposant n est plus grand que m.

Pour cette intégration, M. Euler met en usage la méthode qu'il a employée avec succès dans le mémoire précédent, savoir la résolution en fractions partielles par la décomposition du dénominateur en n facteurs simples trinomiaux de la forme $t^1-2\cos.\omega+t^{-1}$, qui donnent autant de fractions partielles à intégrer, chacune de la forme

$$-\frac{2\sin.\omega\sin.m\omega}{n\cos n\omega}\cdot\frac{\partial t}{t\sqrt{-1}}\cdot\frac{1}{t^1-2\cos.\omega+t^{-1}},$$

les n valeurs de l'angle ω étant $0,\ \frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ldots\frac{(n-1)\pi}{n}$.

Mais comme l'intégration de chacune de ces fractions partielles, quoique très-facile, mène à un arc imaginaire qu'il faudroit réduire à des quantités réelles, pour s'épargner cet embarras, l'Auteur fait rentrer son angle φ dans le calcul, ce qui étant fait, l'intégrale partielle en question, à cause de $\frac{\partial t}{t\sqrt{-1}}=\partial\varphi$ et $t^1+t^{-1}=2\cos.\varphi$, prend cette forme

$$-\frac{\sin.\omega\sin.m\omega}{n\cos.n\omega}\cdot\frac{\partial\varphi}{\cos.\varphi-\cos.\omega},$$

de façon qu'il ne reste plus qu'à trouver l'intégrale de $\frac{\partial\varphi}{\cos.\varphi-\cos.\omega}$, que M. Euler trouve d'une manière très-aisée

$$=\frac{1}{\sin.\omega}\,l\,\frac{\sin.\frac{1}{2}(\omega+\varphi)}{\sin.\frac{1}{2}(\omega-\varphi)}.$$

La forme générale des intégrales partielles qui composent l'intégrale complète de la formule proposée sera donc

$$-\frac{\sin. m\omega}{n\cos. n\omega}\, l\frac{\sin.\frac{1}{2}(\omega+\varphi)}{\sin.\frac{1}{2}(\omega-\varphi)},$$

ou bien, ce qui revient au même, en mettant $\frac{\pi}{n}=2\alpha$ et $\varphi=2\psi$, on aura

$$\int\frac{2\partial\psi\sin. 2m\psi}{\sin. 2n\psi}=\frac{\sin. 2m\alpha}{n}\, l\frac{\sin.(\alpha+\psi)}{\sin.(\alpha-\psi)}-\frac{\sin. 4m\alpha}{n}\, l\frac{\sin.(2\alpha+\psi)}{\sin.(2\alpha-\psi)}$$
$$+\frac{\sin. 6m\alpha}{n}\, l\frac{\sin.(3\alpha+\psi)}{\sin.(3\alpha-\psi)}-\text{etc.},$$

où le nombre des termes est $n-1$ et $\psi<\alpha$.

Ayant donc trouvé une expression finie pour l'intégrale de $\frac{\partial p\sin.\frac{\theta p}{n}}{\sin.\frac{\pi p}{n}}$, on pourra aussi assigner, par une expression finie, l'intégrale de la formule

$$\int\frac{\partial x}{xlx}\cdot\frac{x^p-x^{-p}}{x^n+2\cos.\theta+x^{-n}},$$

toutes les fois que l'angle θ est à π dans un rapport rationnel, c'est à dire $\theta:\pi=\mu:\nu$, ou bien $\nu=\frac{\mu\pi}{\theta}$. En mettant donc $\frac{p}{n}=r$ et $\frac{\pi}{2\nu}=\varrho$, la forme générale de toutes les parties dont l'intégrale de la formule proposée est composée, sera

$$\pm\frac{\sin. i\theta}{\sin.\theta}\, l\frac{\sin.\varrho(i+r)}{\sin.\varrho(i-r)}.$$

Mais comme de cette manière le nombre des termes peut être réduit à la moitié, pour faciliter cette contraction l'Auteur distingue quatre cas, selon que les nombres μ et ν sont tous les deux pairs, ou tous les deux impairs, ou l'un pair et l'autre impair, et il finit son mémoire par quelques exemples propres à éclaircir cette intégration remarquable.

Quanta incrementa Calculo Imaginariorum per universam Analysin accepta sint referenda, nunc quidem amplius nemo dubitabit. Nuper[1]) equidem conatus sum integrationem formularum rationalium a Calculo Imaginariorum penitus liberare; verumtamen hoc negotium in casibus, ubi denominator plures habet factores inter se aequales, minus feliciter successit. Quin etiam non ita pridem in tales formulas integrales incidi, quae quomodo sine subsidio imaginariorum tractari queant, nullo adhuc modo perspicio. Cum enim*)

*) Vide Dissertationem praecedentem p. 284.

1) Vide Commentationem 572 huius voluminis p. 113. A. L.

ostendissem huius formulae integralis

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + 2\cos.\theta + x^{-n}}$$

valorem a termino $x = 0$ usque ad $x = 1$ extensum esse

$$\frac{\pi \sin.\frac{\theta p}{n}}{n \sin.\theta \sin.\frac{\pi p}{n}}$$

denotante π peripheriam circuli, cuius diameter $= 1$, inde facile deducitur haec conclusio maxime memorabilis, quod huius formulae integralis

$$\int \frac{\partial x}{x l x} \cdot \frac{x^p - x^{-p}}{x^n + 2\cos.\theta + x^{-n}}$$

valor pariter a termino $x = 0$ usque ad $x = 1$ extensus aequetur isti integrali

$$\frac{\pi}{n \sin.\theta} \int \frac{\partial p \sin.\frac{\theta p}{n}}{\sin.\frac{\pi p}{n}},$$

ubi scilicet quantitas p tamquam variabilis spectatur et integrale ita capitur, ut evanescat posito $p = 0$. Quodsi ergo nunc faciamus $\frac{p}{n} = \varphi$, integrari oportet huiusmodi formulam differentialem $\frac{\partial\varphi \sin. m\varphi}{\sin. n\varphi}$. Quemadmodum igitur ista integratio auxilio imaginariorum tractari debeat, hic sum ostensurus.

DE INTEGRATIONE FORMULAE

$$\int \frac{\partial\varphi \sin. m\varphi}{\sin. n\varphi}$$

1. Ante omnia hanc formulam ad quantitates algebraicas ordinarias revocari convenit, id quod commodius quam per imaginaria praestari nequit. Hunc in finem statuamus brevitatis gratia

$$t = \cos.\varphi + \sqrt{-1} \cdot \sin.\varphi \quad \text{et} \quad u = \cos.\varphi - \sqrt{-1} \cdot \sin.\varphi,$$

ita ut sit $tu = 1$; tum vero erit

$$\partial t = -\partial\varphi (\sin.\varphi - \sqrt{-1} \cdot \cos.\varphi)$$

ideoque

$$\partial t \sqrt{-1} = -\partial\varphi (\cos.\varphi + \sqrt{-1} \cdot \sin.\varphi) = -t\partial\varphi,$$

unde ergo fiet

$$\partial\varphi = -\frac{\partial t\sqrt{-1}}{t} = \frac{\partial t}{t\sqrt{-1}}.$$

2. His autem formulis constitutis ex elementis Calculi Imaginariorum constat esse

$$t^\lambda = \cos.\lambda\varphi + \sqrt{-1}\cdot\sin.\lambda\varphi \quad \text{et} \quad u^\lambda = \cos.\lambda\varphi - \sqrt{-1}\cdot\sin.\lambda\varphi;$$

unde ergo colligitur $t^\lambda - u^\lambda = 2\sqrt{-1}\cdot\sin.\lambda\varphi$ ideoque

$$\sin.\lambda\varphi = \frac{t^\lambda - u^\lambda}{2\sqrt{-1}}.$$

Hinc ergo, si loco λ scribamus numeros m et n, erit

$$\frac{\sin.m\varphi}{\sin.n\varphi} = \frac{t^m - u^m}{t^n - u^n},$$

quocirca, si integrale quaesitum littera S designemus, ut sit

$$S = \int\frac{\partial\varphi\sin.m\varphi}{\sin.n\varphi},$$

facta substitutione nunc habebimus

$$\partial S = \frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^m - u^m}{t^n - u^n}.$$

Quia autem est $u = \frac{1}{t} = t^{-1}$, formula proposita ad speciem consuetam solam variabilem t involventem est reducta, cum sit

$$\partial S\sqrt{-1} = \frac{\partial t}{t}\cdot\frac{t^m - t^{-m}}{t^n - t^{-n}},$$

cuius formulae adeo integralis iam passim evoluta reperitur. Hic autem probe meminisse oportet ipsam quantitatem t non esse realem, cum sit $t = \cos.\varphi + \sqrt{-1}\cdot\sin.\varphi$.

3. Manifestum hic est ambos numeros m et n semper tamquam integros spectari posse, cum iis ratio indicetur, quam ambo anguli $m\varphi$ et $n\varphi$ inter se tenent. Hic igitur ante omnia dispiciendum erit, utrum exponens m

maior minorve sit exponente n, quandoquidem notum est, si fuerit $m > n$, fractionem nostram esse spuriam atque partes integras ante ex ea elici debere, quam integratio suscipiatur. Hos ergo casus hic primum evolvi conveniet.

Sit igitur primo $m = n + \lambda$, ita tamen, ut sit $\lambda < n$, ac facile patebit fractionem

$$\frac{t^{n+\lambda} - t^{-(n+\lambda)}}{t^n - t^{-n}}$$

continere partem integram $t^\lambda + t^{-\lambda}$, qua ab ista fractione sublata remanet

$$-\frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}},$$

quae fractio non amplius est spuria. Ex parte integra autem ducta in $\frac{\partial t}{t}$ oritur integrale $\frac{t^\lambda - t^{-\lambda}}{\lambda}$. At vero est $t^\lambda - t^{-\lambda} = t^\lambda - u^\lambda = 2\sqrt{-1}\cdot\sin.\lambda\varphi$, quod per $\sqrt{-1}$ divisum dat partem integralis hinc oriundam

$$= \frac{2\sin.\lambda\varphi}{\lambda}.$$

4. Sin autem fuerit $m > 2n$ sive $m = 2n + \lambda$, tum fractio nostra

$$\frac{t^{2n+\lambda} - t^{-2n-\lambda}}{t^n - t^{-n}}$$

hanc continebit partem integram $t^{n+\lambda} + t^{-n-\lambda}$, qua ablata remanet adhuc ista fractio

$$\frac{t^\lambda - t^{-\lambda}}{t^n - t^{-n}},$$

quae iam est genuina ob $\lambda < n$. At vero ex parte integra ducta in $\frac{\partial t}{t}$ oritur integrando $\frac{t^{n+\lambda} - t^{-n-\lambda}}{n+\lambda} = \frac{t^{n+\lambda} - u^{n+\lambda}}{n+\lambda}$, cuius valor est $\frac{2\sqrt{-1}\cdot\sin.(n+\lambda)\varphi}{n+\lambda}$, qui per $\sqrt{-1}$ divisus praebet partem integralis hinc natam

$$= \frac{2\sin.(n+\lambda)\varphi}{n+\lambda}.$$

5. Simili modo si fuerit $m > 3n$ ac ponatur $m = 3n + \lambda$, fractio nostra erit

$$\frac{t^{3n+\lambda} - t^{-3n-\lambda}}{t^n - t^{-n}},$$

quae continebit partem integram $t^{3n+\lambda} + t^{-3n-\lambda}$; hac autem ablata remanebit adhuc fractio

$$\frac{t^{n+\lambda} - t^{-n-\lambda}}{t^n - t^{-n}},$$

quae etiamnunc est spuria et continet partem integram $t^\lambda + t^{-\lambda}$; qua ablata demum remanet fractio genuina

$$-\frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}}.$$

Ex partibus autem integris oriuntur hae partes integralis

$$\frac{2\sin.(2n+\lambda)\varphi}{2n+\lambda} + \frac{2\sin.\lambda\varphi}{\lambda}.$$

6. Ponamus quoque esse $m > 4n$ ideoque $m = 4n + \lambda$ et fractio nostra erit

$$\frac{t^{4n+\lambda} - t^{-4n-\lambda}}{t^n - t^{-n}},$$

quae statim continet partem integram $t^{3n+\lambda} + t^{-3n-\lambda}$; hac autem ablata remanet adhuc ista fractio

$$\frac{t^{2n+\lambda} - t^{-2n-\lambda}}{t^n - t^{-n}},$$

quae denuo continet partem integram $t^{n+\lambda} + t^{-n-\lambda}$; qua subtracta tandem remanet ista fractio genuina

$$\frac{t^\lambda - t^{-\lambda}}{t^n - t^{-n}}.$$

Iam vero ex partibus integris obtinentur pro integrali S istae partes

$$\frac{2\sin.(3n+\lambda)\varphi}{3n+\lambda} + \frac{2\sin.(n+\lambda)\varphi}{n+\lambda}.$$

7. Sit porro etiam $m > 5n$ sive $m = 5n + \lambda$ ac nostra fractio

$$\frac{t^{5n+\lambda} - t^{-5n-\lambda}}{t^n - t^{-n}}$$

primo continebit partem integram $t^{4n+\lambda}+t^{-4n-\lambda}$; qua ablata remanet adhuc ista fractio

$$\frac{t^{3n+\lambda}-t^{-3n-\lambda}}{t^n-t^{-n}},$$

quae per antecedentia continet adhuc duas partes integras, scilicet $t^{2n+\lambda}+t^{-2n-\lambda}$ et $t^{\lambda}+t^{-\lambda}$; quibus ablatis remanet tandem ista fractio genuina

$$\frac{t^{n-\lambda}-t^{-(n-\lambda)}}{t^n-t^{-n}}.$$

8. Ex his casibus iam satis perspicitur, quomodo, si exponens n adhuc maior accipiatur, partes integrae in integrale S ingredientes se sint habiturae, quas idcirco hic coniunctim aspectui exponamus.

I. Si $m=n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.\lambda\varphi}{\lambda}-\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^{n-\lambda}-t^{-(n-\lambda)}}{t^n-t^{-n}}.$$

II. Si $m=2n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(2n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(n+\lambda)\varphi}{n+\lambda}+\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^{\lambda}-t^{-\lambda}}{t^n-t^{-n}}.$$

III. Si $m=3n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(3n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(2n+\lambda)\varphi}{2n+\lambda}+\frac{2\sin.\lambda\varphi}{\lambda}-\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^{n-\lambda}-t^{-(n-\lambda)}}{t^n-t^{-n}}.$$

IV. Si $m=4n+\lambda$, erit

$$\int\frac{\partial\varphi\sin.(4n+\lambda)\varphi}{\sin.n\varphi}$$
$$=\frac{2\sin.(3n+\lambda)\varphi}{3n+\lambda}+\frac{2\sin.(n+\lambda)\varphi}{n+\lambda}+\int\frac{\partial t}{t\sqrt{-1}}\cdot\frac{t^{\lambda}-t^{-\lambda}}{t^n-t^{-n}}.$$

V. Si $m = 5n + \lambda$, erit

$$\int \frac{\partial\varphi \sin.(5n+\lambda)\varphi}{\sin. n\varphi}$$

$$= \frac{2\sin.(4n+\lambda)\varphi}{4n+\lambda} + \frac{2\sin.(2n+\lambda)\varphi}{2n+\lambda} + \frac{2\sin.\lambda\varphi}{\lambda} - \int \frac{\partial t}{t\sqrt{-1}} \cdot \frac{t^{n-\lambda} - t^{-(n-\lambda)}}{t^n - t^{-n}}.$$

VI. Si $m = 6n + \lambda$, erit

$$\int \frac{\partial\varphi \sin.(6n+\lambda)\varphi}{\sin. n\varphi}$$

$$= \frac{2\sin.(5n+\lambda)\varphi}{5n+\lambda} + \frac{2\sin.(3n+\lambda)\varphi}{3n+\lambda} + \frac{2\sin.(n+\lambda)\varphi}{n+\lambda} + \int \frac{\partial t}{t\sqrt{-1}} \cdot \frac{t^{\lambda} - t^{-\lambda}}{t^n - t^{-n}}$$

etc.

9. His igitur casibus, quibus $m > n$, felicissimo cum successu expeditis totum negotium reducitur ad integrationem formulae $\frac{\partial\varphi \sin. m\varphi}{\sin. n\varphi}$ pro casibus, quibus est $m < n$, quandoquidem ex modo allatis manifestum est, quomodo illi casus ad hos facillime reducuntur. Tum igitur ope nostrae substitutionis $t = \cos.\varphi + \sqrt{-1}\cdot\sin.\varphi$ pervenitur ad hanc formulam

$$S\sqrt{-1} = \int \frac{\partial t}{t} \cdot \frac{t^m - t^{-m}}{t^n - t^{-n}},$$

cuius ergo integrationem data opera instituamus.

INVESTIGATIO INTEGRALIS $\int \frac{\partial t}{t} \cdot \frac{t^m - t^{-m}}{t^n - t^{-n}}$ EXISTENTE $m < n$

10. Hic ante omnia cuncti factores trinomiales nostri denominatoris $t^n - t^{-n}$ indagari debebunt, quorum singulorum forma ita exhiberi potest $t^1 - 2\cos.\omega + t^{-1}$, ubi angulum ω ita definiri oportet, ut posito

$$t^1 - 2\cos.\omega + t^{-1} = 0$$

simul ipse denominator evanescat; tum autem exinde colligitur

$$t = \cos.\omega + \sqrt{-1}\cdot\sin.\omega,$$

unde statim patet fore

$$t^n = \cos. n\omega + \sqrt{-1} \cdot \sin. n\omega \quad \text{et} \quad t^{-n} = \cos. n\omega - \sqrt{-1} \cdot \sin. n\omega,$$

quamobrem noster denominator reducetur ad hanc formam $2\sqrt{-1} \cdot \sin. n\omega$, qui ergo valor nihilo debet aequari.

11. Cum igitur debeat esse $\sin. n\omega = 0$, omnes valores, quos pro $n\omega$ accipere licet, erunt 0π, π, 2π, 3π etc., unde ipsius anguli ω valores erunt $\frac{0\pi}{n}$, $\frac{1\pi}{n}$, $\frac{2\pi}{n}$, $\frac{3\pi}{n}$ etc. et in genere $\frac{i\pi}{n}$ denotante i numerum integrum quemcunque. Hinc igitur pro omnibus factoribus nostri denominatoris videntur capi debere n horum valorum; verum manifestum est, quotcunque tales formulae $t^1 - 2\cos. \omega + t^{-1}$ in se invicem multiplicentur, ultimum terminum nunquam prodire posse $-t^{-n}$. At vero hic meminisse oportet, quae circa huiusmodi integrationes in genere sunt praecepta, scilicet talem factorem trinomialem $tt - 2t\cos. \omega + 1$ casu, quo $\omega = 0$, non factorem quadratum $(t-1)^2$, sed tantum simplicem $t-1$ innui, quod idem quoque evenit, si $\omega = \pi$; tum enim quoque non factor quadratus $(t+1)^2$, sed tantum simplex $t+1$ est sumendus; quare cum hi ipsi casus inter valores ipsius ω occurrant, necesse est, ut numerus horum factorum unitate augeatur. Hic autem commode usu venit, ut isti casus ex valoribus $\omega = 0$ et $\omega = \pi$ oriundi e medio tollantur.

12. Cum igitur fractionem nostram $\frac{t^m - t^{-m}}{t^n - t^{-n}}$ in meras fractiones simplices resolvi oporteat, quarum denominatores sint $t^1 - 2\cos. \omega + t^{-1}$, pro unaquaque harum fractionum statuamus

$$\frac{t^m - t^{-m}}{t^n - t^{-n}} = \frac{\varDelta}{t - 2\cos. \omega + t^{-1}} + R,$$

ubi R complectatur omnes reliquas fractiones, et nunc utrinque multiplicemus per $t - 2\cos. \omega + t^{-1}$, ut prodeat

$$\frac{(t^m - t^{-m})(t - 2\cos. \omega + t^{-1})}{t^n - t^{-n}} = \varDelta + R(t - 2\cos. \omega + t^{-1});$$

unde si iam ponamus $t - 2\cos. \omega + t^{-1} = 0$, quod fit sumendo

$$t = \cos. \omega + \sqrt{-1} \cdot \sin. \omega,$$

hinc colligitur numerator nostrae fractionis

$$\varDelta = (t^m - t^{-m})\frac{t - 2\cos.\omega + t^{-1}}{t^n - t^{-n}}.$$

Tum autem manifestum est in hac fractione, ad quam sumus deducti, hoc casu tam numeratorem quam denominatorem in nihilum abire, unde iuxta regulam notissimam eorum loco sua scribamus differentialia, atque ista fractio induet hanc formam $\frac{t^1 - t^{-1}}{n(t^n + t^{-n})}$, ubi manifesto erit $t^1 - t^{-1} = 2\sqrt{-1}\cdot\sin.\omega$, at $t^n + t^{-n} = 2\cos.n\omega$, ita ut nunc valor huius fractionis futurus sit $\frac{\sqrt{-1}\cdot\sin.\omega}{n\cos.n\omega}$, qui ductus in $t^m - t^{-m} = 2\sqrt{-1}\cdot\sin.m\omega$ dabit numeratorem nostrum quaesitum

$$\varDelta = -\frac{2\sin.\omega\sin.m\omega}{n\cos.n\omega}.$$

Quia autem est $\sin.n\omega = 0$, semper erit vel $\cos.n\omega = 1$ vel $\cos.n\omega = -1$, prouti statuendo in genere $\omega = \frac{i\pi}{n}$ numerus i fuerit vel par vel impar.

13. Inventa igitur hac fractione

$$-\frac{2\sin.\omega\sin.m\omega}{n\cos.n\omega}\cdot\frac{1}{t - 2\cos.\omega + t^{-1}}$$

ea in $\frac{\partial t}{t}$ multiplicetur et integretur sicque ad istam pertingimus formulam integralem

$$-\frac{2\sin.\omega\sin.m\omega}{n\cos.n\omega}\int\frac{\partial t}{t}\cdot\frac{1}{t - 2\cos.\omega + t^{-1}},$$

cuius quidem integratio nulla amplius laborat difficultate; perduceret enim ad arcum circuli, cuius tangens $= \frac{t\sin.\omega}{1 - t\cos.\omega}$; verum quia ipsa quantitas t iam est imaginaria, hinc parum lucraremur, quoniam necesse foret istum arcum imaginarium ad quantitates reales reducere, siquidem constat arcus imaginarios ad logarithmos reales reduci.

14. Ut igitur hunc laborem evitemus, loco nostrae variabilis t ipsum angulum φ rursus in calculum revocemus, et quia iam vidimus esse $\frac{\partial t}{t} = \partial\varphi\sqrt{-1}$, tum vero $t + u = 2\cos.\varphi$, hisce valoribus substitutis formula

integranda erit

$$-\frac{\sin.\omega\sin.m\omega}{n\cos.n\omega}\cdot\frac{\partial\varphi\sqrt{-1}}{\cos.\varphi-\cos.\omega},$$

quae formula per $\sqrt{-1}$ divisa praebet partem ipsius integralis quaesiti S, ita ut sit S aggregatum omnium harum formularum

$$-\frac{\sin.\omega\sin.m\omega}{n\cos.n\omega}\int\frac{\partial\varphi}{\cos.\varphi-\cos.\omega},$$ [1)]

isquidem angulo ω successive omnes suos valores tribuamus; ubi per se manifestum est in hac integratione angulum ω esse constantem solumque φ variabilem.

15. Ex coefficiente huius formulae statim patet, quod iam supra innuimus, ex valoribus ipsius ω primo et extremo, scilicet $\omega=0$ et $\omega=\pi$, partes integralis sponte e medio tolli, ita ut nunc sufficiat loco ω successive substitui hos valores $\frac{\pi}{n}$, $\frac{2\pi}{n}$, $\frac{3\pi}{n}$, ... $\frac{(n-1)\pi}{n}$. Ubi recordandum, dum statuitur $\omega=\frac{i\pi}{n}$, quoties i fuerit numerus par, fore $\cos.n\omega=+1$; sin autem sit i numerus impar, tum fore $\cos.n\omega=-1$. Quibus observatis totum negotium reductum est ad integrationem huius formulae satis memorabilis $\int\frac{\partial\varphi}{\cos.\varphi-\cos.\omega}$.

16. Facile quidem foret istam formulam ad quantitates reales consuetas nevocare; interim tamen sequenti modo haec integratio facilius et elegantius rabsolvi potest. Ponamus enim brevitatis gratia $\frac{\partial\varphi}{\cos.\varphi-\cos.\omega}=\partial s$ et secundum calculum angulorum iam satis vulgatum novimus esse

$$\cos.\varphi-\cos.\omega=2\sin.\frac{\omega+\varphi}{2}\sin.\frac{\omega-\varphi}{2}$$

sicque habebimus

$$\partial s=\frac{\partial\varphi}{2\sin.\frac{\omega+\varphi}{2}\sin.\frac{\omega-\varphi}{2}}$$

sive

$$\frac{2\partial s}{\partial\varphi}=\frac{1}{\sin.\frac{\omega+\varphi}{2}\sin.\frac{\omega-\varphi}{2}},$$

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1) **Editio princeps:** ... *quaesiti S, ita ut sit*

$$S=-\frac{\sin.\omega\sin.m\omega}{n\cos.n\omega}\int\frac{\partial\varphi}{\cos.\varphi-\cos.\omega}.$$

A. L.

quae fractio, quia denominator duobus constat factoribus, commode resolvi potest in duas fractiones huiusmodi

$$\frac{\alpha \cos. \frac{\omega+\varphi}{2}}{\sin. \frac{\omega+\varphi}{2}} + \frac{\beta \cos. \frac{\omega-\varphi}{2}}{\sin. \frac{\omega-\varphi}{2}},$$

ubi statim patet sumi debere $\beta = \alpha$; tum enim summa harum fractionum prodit

$$\frac{\alpha \sin. \omega}{\sin. \frac{\omega+\varphi}{2} \sin. \frac{\omega-\varphi}{2}},$$

unde

$$\alpha = \beta = \frac{1}{\sin. \omega}.$$

Hinc autem erit

$$\partial s = \frac{1}{2 \sin. \omega} \left(\frac{\partial \varphi \cos. \frac{\omega+\varphi}{2}}{\sin. \frac{\omega+\varphi}{2}} + \frac{\partial \varphi \cos. \frac{\omega-\varphi}{2}}{\sin. \frac{\omega-\varphi}{2}} \right),$$

in quibus formulis numerator manifesto est differentiale denominatoris, unde concludimus fore

$$s = \frac{1}{\sin. \omega} l \frac{\sin. \frac{\omega+\varphi}{2}}{\sin. \frac{\omega-\varphi}{2}}.$$

17. Invento iam hoc integrali, in quo cardo totius investigationis versabatur, quilibet factor denominatoris in valorem integralem quaesitum S ductus suppeditat istam partem

$$-\frac{\sin. m\omega}{n \cos. n\omega} l \frac{\sin. \frac{\omega+\varphi}{2}}{\sin. \frac{\omega-\varphi}{2}},$$

ubi tantum opus est, ut loco anguli ω successive omnes eius valores debiti substituantur; tum enim aggregatum omnium harum formularum praebebit verum valorem integralis

$$S = \int \frac{\partial \varphi \sin. m\varphi}{\sin. n\varphi}.$$

18. Quo autem totum integrale succinctius repraesentare valeamus, ponamus brevitatis gratia $\frac{\pi}{n} = 2\alpha$, ita ut valores ipsius ω futuri sint 2α, 4α,

$6\alpha, \ldots 2(n-1)\alpha$; tum vero sit $\varphi = 2\psi$ atque formulae integralis

$$\int \frac{2\partial\psi \sin. 2m\psi}{\sin. 2n\psi}$$

valor completus erit

$$\begin{aligned} S = {} & \frac{\sin. 2m\alpha}{n} l \frac{\sin. (\alpha+\psi)}{\sin. (\alpha-\psi)} - \frac{\sin. 4m\alpha}{n} l \frac{\sin. (2\alpha+\psi)}{\sin. (2\alpha-\psi)} \\ & + \frac{\sin. 6m\alpha}{n} l \frac{\sin. (3\alpha+\psi)}{\sin. (3\alpha-\psi)} - \frac{\sin. 8m\alpha}{n} l \frac{\sin. (4\alpha+\psi)}{\sin. (4\alpha-\psi)} \\ & + \frac{\sin. 10m\alpha}{n} l \frac{\sin. (5\alpha+\psi)}{\sin. (5\alpha-\psi)} - \frac{\sin. 12m\alpha}{n} l \frac{\sin. (6\alpha+\psi)}{\sin. (6\alpha-\psi)} \\ & \text{etc.}, \end{aligned}$$

donec horum membrorum numerus sit $n-1$. Haec autem formula tantum valet, quando $m < n$; si enim fuerit $m > n$, iam ante ostendimus, cuiusmodi termini insuper debeant adiungi.

19. Hic observandum est haec integralia ita esse sumta, ut evanescant posito $\varphi = 0$, quoniam hoc casu omnes logarithmi ad unitatem referuntur. Deinde etiam evidens est, si angulus ψ augeatur usque ad α, tum integrale iam in infinitum excrescere; unde patet hunc angulum non ultra istum terminum augeri convenire. Verum etiam casus initio memoratus, qui ad hanc formulam integralem ducit, non postulat, ut iste angulus ultra hunc terminum augeatur, quamobrem operae pretium erit integrationem inventam ad hunc ipsum casum accommodare.

PROBLEMA

Valorem istius formulae integralis

$$\int \frac{\partial x}{x l x} \cdot \frac{x^p - x^{-p}}{x^n + 2\cos.\theta + x^{-n}}$$

a termino $x = 0$ *ad* $x = 1$ *extensum per expressionem finitam assignare.*

SOLUTIO

20. Quoniam istum valorem quaesitum reduxi ad hanc formulam integralem

$$\frac{\pi}{n\sin.\theta}\int\frac{\partial p\sin.\frac{\theta p}{n}}{\sin.\frac{\pi p}{n}},$$

primum tenendum est eum finite exprimi non posse, nisi angulus θ ad π habeat rationem rationalem. Ponamus ergo hanc rationem esse $\theta:\pi=\mu:\nu$, ita ut μ et ν sint numeri integri, quamobrem pro formula ante tractata statuamus $m=\mu$ et $n=\nu$, unde fiet angulus $\nu\varphi=\frac{\pi p}{n}$. Ponamus hic brevitatis gratia $\frac{p}{n}=r$, ut habeamus $\varphi=\frac{\pi r}{\nu}$, et valor, quem quaerimus, ob $p=nr$ erit

$$\frac{\pi}{\sin.\theta}\int\frac{\partial r\sin.\theta r}{\sin.\pi r};$$

quare, cum hinc fiat $\varphi=\frac{\pi r}{\nu}$, formula supra tractata $\int\frac{\partial\varphi\sin.m\varphi}{\sin.n\varphi}$ abibit in hanc

$$S=\frac{\pi}{\nu}\int\frac{\partial r\sin.\frac{\mu\pi r}{\nu}}{\sin.\pi r}$$

sicque valor, quem hic quaerimus, erit $\frac{\nu S}{\sin.\theta}$, ita ut tantum opus sit valorem ipsius S pro hoc casu evolvere.

21. Consideremus nunc primum valorem ipsius ω, qui erat $\omega=\frac{\pi}{n}=\frac{\pi}{\nu}$, qui pro S produxit partem integralem

$$-\frac{\sin.m\omega}{n\cos.n\omega}\,l\frac{\sin.\frac{\omega+\varphi}{2}}{\sin.\frac{\omega-\varphi}{2}};$$

erit hic $m\omega=\frac{\mu\pi}{\nu}=\theta$ et $\cos.n\omega=-1$; tum vero

$$\omega+\varphi=\frac{\pi}{\nu}(1+r)\quad\text{et}\quad\omega-\varphi=\frac{\pi}{\nu}(1-r).$$

Primum igitur hic sumi debet angulus $\frac{\pi}{2\nu}$, quem brevitatis gratia ponamus $=\varrho$, ut sit $\varrho=\frac{\pi}{2\nu}$, et prima pars nostrae formulae S erit

$$\frac{\sin.\theta}{\nu}\,l\frac{\sin.\varrho(1+r)}{\sin.\varrho(1-r)},$$

sequentes autem partes erunt

$$-\frac{\sin. 2\theta}{\nu} l \frac{\sin. \varrho(2+r)}{\sin. \varrho(2-r)} + \frac{\sin. 3\theta}{\nu} l \frac{\sin. \varrho(3+r)}{\sin. \varrho(3-r)} - \text{etc.},$$

quae partes ductae in $\frac{\nu}{\sin. \theta}$ praebent ipsum valorem, quem nostrum problema postulat, qui ergo erit

$$\begin{aligned} &\frac{\sin. \theta}{\sin. \theta} l \frac{\sin. \varrho(1+r)}{\sin. \varrho(1-r)} - \frac{\sin. 2\theta}{\sin. \theta} l \frac{\sin. \varrho(2+r)}{\sin. \varrho(2-r)} \\ &+ \frac{\sin. 3\theta}{\sin. \theta} l \frac{\sin. \varrho(3+r)}{\sin. \varrho(3-r)} - \frac{\sin. 4\theta}{\sin. \theta} l \frac{\sin. \varrho(4+r)}{\sin. \varrho(4-r)} \\ &\text{etc.}; \end{aligned}$$

quae membra eousque continuari debent, donec eorum numerus fiat $\nu - 1$, ubi pro nostro problemate tantum notetur esse $r = \frac{p}{n}$ et $\varrho = \frac{\pi}{2\nu}$ existente $\theta : \pi = \mu : \nu$ sive $\theta = \frac{\mu\pi}{\nu}$, ita ut μ sit numerus integer. Cum igitur in formula proposita exponens p necessario minor sit quam n, erit r unitate minor ideoque omnes istae formulae finitae.

22. Forma igitur generalis omnium partium, ex quibus hoc integrale constat, est

$$\pm \frac{\sin. i\theta}{\sin. \theta} l \frac{\sin. \varrho(i+r)}{\sin. \varrho(i-r)},$$

ubi signum superius $+$ valet, quoties i fuerit numerus impar, inferius vero $-$, si par. Pro ultima igitur harum partium erit $i = \nu - 1$. Ubi probe notetur, si sumeremus $i = \nu$, partem hinc resultantem sponte esse evanituram, propterea quod $i\varrho = \nu\varrho = \frac{\pi}{2}$ ideoque ambo sinus post logarithmum inter se aequales, ita ut perinde sit, sive membrorum numerus statuatur $= \nu - 1$ sive $= \nu$.

23. Consideremus nunc ultimum membrum nostri valoris integralis sumendo $i = \nu - 1$, unde fiet $\sin. (\nu - 1)\theta = \sin. (\mu\pi - \theta)$, qui erit $= \sin. \theta$, si μ fuerit numerus impar; sin autem μ fuerit numerus par, is erit $= -\sin. \theta$. Tum vero erit

$$i\varrho = (\nu - 1)\varrho = \frac{\pi}{2} - \frac{\pi}{2\nu}$$

ideoque

$$\sin.\varrho(\nu - 1 + r) = \sin.\left(\frac{\pi}{2} - \varrho(1 - r)\right) = \cos.\varrho(1 - r).$$

Simili modo pro denominatore erit

$$\sin.\varrho(i - r) = \sin.\left(\frac{\pi}{2} - \varrho(1 + r)\right) = \cos.\varrho(1 + r),$$

ita ut in ultimo membro cosinus eorundem angulorum occurrant, quorum sinus occurrunt in primo membro, quae permutatio etiam reperietur in membro penultimo et secundo, tum vero etiam in antepenultimo et tertio, unde bina huiusmodi membra in unum coniungi poterunt.

24. Hic autem quatuor casus examinari convenit, prouti ambo numeri μ et ν fuerint numeri vel pares vel impares.

Sint igitur primo ambo pares, unde coefficiens ultimi membri erit $\frac{\sin.(\mu\pi - \theta)}{\sin.\theta} = -\frac{\sin.\theta}{\sin.\theta}$ ideoque totum membrum ultimum $= -\frac{\sin.\theta}{\sin.\theta}\,l\frac{\cos.\varrho(1-r)}{\cos.\varrho(1+r)}$, quamobrem primum membrum cum ultimo coniunctum dabit

$$\frac{\sin.\theta}{\sin.\theta}\,l\frac{\sin.\varrho(1+r)}{\sin.\varrho(1-r)}\cdot\frac{\cos.\varrho(1+r)}{\cos.\varrho(1-r)} = \frac{\sin.\theta}{\sin.\theta}\,l\frac{\sin.2\varrho(1+r)}{\sin.2\varrho(1-r)}.$$

Simili modo secundum membrum et penultimum coalescent in

$$-\frac{\sin.2\theta}{\sin.\theta}\,l\frac{\sin.2\varrho(2+r)}{\sin.2\varrho(2-r)};$$

tum vero etiam membrum tertium cum antepenultimo dabit

$$\frac{\sin.3\theta}{\sin.\theta}\,l\frac{\sin.2\varrho(3+r)}{\sin.2\varrho(3-r)}.$$

Sicque de ceteris, ita ut hoc modo numerus membrorum ad semissem reducatur.

25. Maneat nunc ν numerus par, sit vero μ numerus impar eritque coefficiens ultimi membri $\frac{\sin.\theta}{\sin.\theta}$, quod ergo cum primo coniunctum dabit

$$\frac{\sin.\theta}{\sin.\theta}\,l\frac{\sin.\varrho(1+r)}{\sin.\varrho(1-r)}\cdot\frac{\cos.\varrho(1-r)}{\cos.\varrho(1+r)} = \frac{\sin.\theta}{\sin.\theta}\,l\frac{\text{tang}.\varrho(1+r)}{\text{tang}.\varrho(1-r)}.$$

Eodem modo membrum secundum cum penultimo contrahetur in hanc formam

$$-\frac{\sin. 2\theta}{\sin. \theta} l \frac{\operatorname{tang}. \varrho (2+r)}{\operatorname{tang}. \varrho (2-r)};$$

at tertium membrum cum antepenultimo coniunctum dabit

$$\frac{\sin. 3\theta}{\sin. \theta} l \frac{\operatorname{tang}. \varrho (3+r)}{\operatorname{tang}. \varrho (3-r)}.$$

26. Sit nunc ν numerus impar, at μ numerus par et ob priorem conditionem coefficiens ultimi termini erit $-\frac{\sin. (\mu\pi - \theta)}{\sin. \theta}$, qui ob μ numerum parem fiet $\frac{\sin. \theta}{\sin. \theta}$ ideoque uti in casu secundo, unde etiam primum membrum cum ultimo iunctum dabit

$$\frac{\sin. \theta}{\sin. \theta} l \frac{\operatorname{tang}. \varrho (1+r)}{\operatorname{tang}. \varrho (1-r)};$$

secundum vero cum penultimo iunctum

$$-\frac{\sin. 2\theta}{\sin. \theta} l \frac{\operatorname{tang}. \varrho (2+r)}{\operatorname{tang}. \varrho (2-r)};$$

tum vero etiam tertium cum antepenultimo iunctum dat

$$\frac{\sin. 3\theta}{\sin. \theta} l \frac{\operatorname{tang}. \varrho (3+r)}{\operatorname{tang}. \varrho (3-r)}.$$

27. Sint denique ambo numeri μ et ν impares atque evidens est hunc casum ad primum esse rediturum ideoque primum et ultimum membrum contrahi in

$$\frac{\sin. \theta}{\sin. \theta} l \frac{\sin. 2\varrho (1+r)}{\sin. 2\varrho (1-r)},$$

secundum et penultimum in

$$-\frac{\sin. 2\theta}{\sin. \theta} l \frac{\sin. 2\varrho (2+r)}{\sin. 2\varrho (2-r)},$$

tertium et antepenultimum in

$$\frac{\sin. 3\theta}{\sin. \theta} l \frac{\sin. 2\varrho (3+r)}{\sin. 2\varrho (3-r)}.$$

Unde patet hos quatuor casus ad duos reduci posse, prouti ambo numeri μ et ν fuerint vel eiusdem indolis, scilicet ambo vel pares vel impares, vel diversae indolis, alter par, alter impar. Priore casu eadem contractio locum habebit, quam casu primo dedimus, posteriore vero, quam pro secundo dedimus.

28. Ex his intelligitur, si numerus ν fuerit impar ideoque numerus membrorum primum inventorum $\nu-1$ par, tum omnia illa membra contrahi in numerum duplo minorem, scilicet $\frac{\nu-1}{2}$. At vero si ν fuerit numerus par, ob $\nu-1$ imparem facta illa contractione remanebit unum membrum medium respondens valori $i=\frac{\nu}{2}$, pro quo iste reperietur logarithmus

$$l\frac{\sin.\varrho\left(\frac{\nu}{2}+r\right)}{\sin.\varrho\left(\frac{\nu}{2}-r\right)}=l\frac{\sin.\left(\frac{\pi}{4}+\varrho r\right)}{\sin.\left(\frac{\pi}{4}-\varrho r\right)}.$$

Quia igitur est $\sin.\left(\frac{\pi}{4}-\varrho r\right)=\cos.\left(\frac{\pi}{4}+\varrho r\right)$, evidens est hoc casu haberi

$$l\operatorname{tang.}\left(\frac{\pi}{4}+\varrho r\right);$$

coefficiens autem erit

$$\pm\frac{\sin.\frac{\nu}{2}\theta}{\sin.\theta},$$

ubi signum superius valebit, si $\frac{\nu}{2}$ fuerit impar, inferius vero, si par. Est vero $\sin.\frac{\nu}{2}\theta=\sin.\frac{\mu\pi}{2}$, unde patet, si fuerit μ numerus par, hoc membrum penitus e medio tolli; sin autem μ fuerit numerus impar, tum $\sin.\frac{\mu\pi}{2}$ erit vel $+1$ vel -1. Ista ambiguitas autem iam ante est sublata.

His notatis sequentia exempla simpliciora percurramus; ubi notasse iuvabit numerum μ semper minorem esse debere quam ν neque tamen sumi posse $\mu=0$.

29. Quo autem evolutionem casuum specialium faciliorem reddamus, denotet Σ formulam illam integralem, cuius valorem hactenus per partes evolvimus, ita ut sit

$$\Sigma=\frac{\pi}{\sin.\theta}\int\frac{\partial r\sin.\theta r}{\sin.\pi r};$$

tum igitur duos casus distingui conveniet, prouti ambo numeri μ et ν fuerint eiusdem vel diversae indolis.

I. Sint μ et ν eiusdem indolis eritque

$$\begin{aligned}\Sigma = &\frac{\sin.\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(1+r)}{\sin.2\varrho(1-r)} - \frac{\sin.2\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(2+r)}{\sin.2\varrho(2-r)}\\ &+\frac{\sin.3\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(3+r)}{\sin.2\varrho(3-r)} - \frac{\sin.4\theta}{\sin.\theta}\, l\frac{\sin.2\varrho(4+r)}{\sin.2\varrho(4-r)}\\ &\text{etc.,}\end{aligned}$$

quas formulas non ultra multitudinem $\frac{\nu-1}{2}$ continuari necesse est; neque enim hic terminus medius locum habet; si enim fuerit ν numerus par, erit etiam μ par ideoque termini medii coefficiens evanescit.

II. Sint numeri μ et ν diversae indolis vidimusque fore

$$\begin{aligned}\Sigma = &\frac{\sin.\theta}{\sin.\theta}\, l\frac{\operatorname{tang}.\varrho(1+r)}{\operatorname{tang}.\varrho(1-r)} - \frac{\sin.2\theta}{\sin.\theta}\, l\frac{\operatorname{tang}.\varrho(2+r)}{\operatorname{tang}.\varrho(2-r)}\\ &+\frac{\sin.3\theta}{\sin.\theta}\, l\frac{\operatorname{tang}.\varrho(3+r)}{\operatorname{tang}.\varrho(3-r)} - \frac{\sin.4\theta}{\sin.\theta}\, l\frac{\operatorname{tang}.\varrho(4+r)}{\operatorname{tang}.\varrho(4-r)}\\ &\text{etc.,}\end{aligned}$$

quos terminos non ultra multitudinem $\frac{\nu-1}{2}$ continuari oportet. Hic autem, quoties ν numerus par ideoque μ impar, occurret terminus medius, qui nunc ultimum locum occupabit eritque

$$\pm\frac{1}{\sin.\theta}\, l\operatorname{tang}.\left(\frac{\pi}{4}+\varrho r\right),$$

ubi signorum ambiguitas sequitur alternationem signorum. Ceterum hic ubique recordandum est esse $\varrho = \frac{\pi}{2\nu}$ et $\theta = \frac{\mu\pi}{\nu}$.

EXEMPLUM 1 QUO $\nu = 2$

30. Hic igitur erit $\varrho = \frac{\pi}{4} = 45^0$, at numerus μ necessario est $= 1$. Quia igitur $\frac{\nu-1}{2} = \frac{1}{2}$, hic solus terminus, quem medium vocamus, occurrit, ita ut nunc habeamus

$$\Sigma = l\operatorname{tang}.\frac{\pi}{4}(1+r) = l\operatorname{tang}.45^0(1+r),$$

qui valor sponte ex forma generali deducitur, cum sit

$$\Sigma = \pi\int\frac{\partial r\sin.\frac{\pi r}{2}}{\sin.\pi r};$$

est vero $\sin.\pi r = 2\sin.\frac{\pi r}{2}\cos.\frac{\pi r}{2}$, unde fit

$$\Sigma = \frac{\pi}{2}\int\frac{\partial r}{\cos.\frac{\pi r}{2}}.$$

Quodsi iam ponamus $\frac{\pi r}{2} = \varphi$, ob $\frac{\pi\partial r}{2} = \partial\varphi$ erit

$$\Sigma = \int\frac{\partial\varphi}{\cos.\varphi} = l\,\mathrm{tang.}\left(45^0 + \frac{1}{2}\varphi\right).$$

Restituto ergo pro φ valore assumto erit $\Sigma = l\,\mathrm{tang.}\,45^0(1+r)$, uti invenimus.

EXEMPLUM 2 QUO $\nu = 3$

31. Hic ergo erit $\varrho = \frac{\pi}{6} = 30^0$, et quia $\frac{\nu-1}{2} = 1$, integrale nostrum unico constabit termino. Nunc autem numerus μ duos valores habere potest, 1 et 2. Sit primo $\mu = 1$ hincque $\theta = \frac{\pi}{3} = 60^0$, et quia ambo numeri sunt impares, ex casu primo colligimus

$$\Sigma = l\,\frac{\sin.60^0(1+r)}{\sin.60^0(1-r)}.$$

At si fuerit $\mu = 2$ ideoque $\theta = 120^0$, quia numeri μ et ν habent disparia signa, ex casu secundo habebimus

$$\Sigma = l\,\frac{\mathrm{tang.}\,30^0(1+r)}{\mathrm{tang.}\,30^0(1-r)}.$$

EXEMPLUM 3 QUO $\nu = 4$

32. Hic ergo erit $\varrho = \frac{\pi}{8} = 22\frac{1}{2}^0$, et quia $\frac{\nu-1}{2} = 1\frac{1}{2}$, integrale unico tantum membro integro constabit, nisi forte terminus medius accedat, quemadmodum singulis casibus pro μ assumtis videbimus.

1°. Sit igitur $\mu = 1$; erit $\theta = 45^0$ et $2\theta = 90^0$. Hinc ergo ob numeros μ et ν dispares ex casu secundo habebimus

$$\Sigma = l\,\frac{\mathrm{tang.}\,22\frac{1}{2}^0(1+r)}{\mathrm{tang.}\,22\frac{1}{2}^0(1-r)} - \sqrt{2}\cdot l\,\mathrm{tang.}\,22\tfrac{1}{2}^0(2+r).$$

2°. Sit $\mu = 2$ eritque $\theta = 90^0$ et $2\theta = 180^0$. Hinc ex casu primo nanciscimur

$$\Sigma = l\,\frac{\sin. 45^0(1+r)}{\sin. 45^0(1-r)}.$$

Cum autem sit $\sin. 45^0(1-r) = \cos. 45^0(1+r)$, evidens est fore

$$\Sigma = l\,\mathrm{tang.}\, 45^0(1+r),$$

quia casus utique convenit cum ratione $\mu : \nu = 1 : 2$.

3°. At si $\mu = 3$ ideoque $\theta = 135^0$ et $2\theta = 270^0$, cuius anguli sinus est -1, ob signa disparia habebimus ex casu secundo

$$\Sigma = l\,\frac{\mathrm{tang.}\, 22\tfrac{1}{2}^0(1+r)}{\mathrm{tang.}\, 22\tfrac{1}{2}^0(1-r)} + \sqrt{2}\cdot l\,\mathrm{tang.}\, 22\tfrac{1}{2}^0(2+r).$$

EXEMPLUM 4 QUO $\nu = 5$

33. Hic ergo erit $\varrho = 18^0$, et quia $\frac{\nu-1}{2} = 2$, integralia ex duobus membris integris constabunt, quia terminus medius, quem quasi dimidium spectamus, hic non occurrit.

1°. Sit $\mu = 1$ eritque $\theta = 36^0$ et $2\theta = 72^0$; hinc ob ambo signa eadem casus primus nobis dat

$$\Sigma = l\,\frac{\sin. 36^0(1+r)}{\sin. 36^0(1-r)} - \frac{\sin. 72^0}{\sin. 36^0}\,l\,\frac{\sin. 36^0(2+r)}{\sin. 36^0(2-r)}.$$

2°. Sit $\mu = 2$ eritque $\theta = 72^0$ ideoque $\sin. 2\theta = \sin. 36^0$; unde ob signa disparia casus secundus dat

$$\Sigma = l\,\frac{\mathrm{tang.}\, 18^0(1+r)}{\mathrm{tang.}\, 18^0(1-r)} - \frac{\sin. 36^0}{\sin. 72^0}\,l\,\frac{\mathrm{tang.}\, 18^0(2+r)}{\mathrm{tang.}\, 18^0(2-r)}.$$

3°. Sit $\mu = 3$ ideoque $\theta = 108^0$ sive $\sin. \theta = \sin. 72^0$ et $\sin. 2\theta = -\sin. 36^0$; unde ob signa paria casus primus dat

$$\Sigma = l\,\frac{\sin. 36^0(1+r)}{\sin. 36^0(1-r)} + \frac{\sin. 36^0}{\sin. 72^0}\,l\,\frac{\sin. 36^0(2+r)}{\sin. 36^0(2-r)}.$$

4°. Sit denique $\mu=4$ et $\theta=144^0$ hincque $\sin.\theta=\sin.36^0$ et $\sin.2\theta=-\sin.72^0$; unde ob signa disparia casus secundus praebet

$$\Sigma = l\frac{\text{tang.}\,18^0(1+r)}{\text{tang.}\,18^0(1-r)} + \frac{\sin.\,72^0}{\sin.\,36^0}\,l\frac{\text{tang.}\,18^0(2+r)}{\text{tang.}\,18^0(2-r)}.$$

EXEMPLUM 5 QUO $\nu=6$

34. Hic igitur est $\varrho=\frac{\pi}{12}=15^0$, et quia $\frac{\nu-1}{2}=\frac{5}{2}$, integralia duobus membris integris constabunt, quibus accedere potest terminus medius sive membrum dimidium, quando scilicet μ est numerus impar.

1°. Sit $\mu=1$; erit $\theta=\frac{\pi}{6}=30^0$, hinc $\sin.\theta=\frac{1}{2}$, $\sin.2\theta=\frac{\sqrt{3}}{2}$ et $\sin.3\theta=1$; quare ob signa disparia secundus casus nobis suppeditat

$$\Sigma = l\frac{\text{tang.}\,15^0(1+r)}{\text{tang.}\,15^0(1-r)} - \sqrt{3}\cdot l\frac{\text{tang.}\,15^0(2+r)}{\text{tang.}\,15^0(2-r)} + 2\,l\,\text{tang.}\,15^0(3+r).$$

2°. Sit $\mu=2$ ideoque $\theta=60^0$, unde fit $\sin.\theta=\frac{\sqrt{3}}{2}$, $\sin.2\theta=\frac{\sqrt{3}}{2}$ et $\sin.3\theta=0$; unde ob signa paria ex casu primo colligimus

$$\Sigma = l\frac{\sin.\,30^0(1+r)}{\sin.\,30^0(1-r)} - l\frac{\sin.\,30^0(2+r)}{\sin.\,30^0(2-r)},$$

quae expressio perfecte aequalis prodiit ei, quam supra invenimus pro casu $\nu=3$ et $\mu=1$.

3°. Sit $\mu=3$ ideoque $\theta=90^0$, hinc $\sin.\theta=1$, $\sin.2\theta=0$ et $\sin.3\theta=-1$; unde ob signa disparia casus secundus nobis praebet

$$\Sigma = l\frac{\text{tang.}\,15^0(1+r)}{\text{tang.}\,15^0(1-r)} + * - l\,\text{tang.}\,15^0(3+r)$$

sive

$$\Sigma = l\frac{\text{tang.}\,15^0(1+r)}{\text{tang.}\,15^0(1-r)\,\text{tang.}\,15^0(3+r)},$$

quae expressio aequalis esse debet ei, quae in primo exemplo prodiit, quia utroque casu est $\mu:\nu=1:2$.

4°. Sit $\mu=4$ ideoque $\theta=120°$, hinc $\sin.\theta=\frac{\sqrt{3}}{2}$, $\sin.2\theta=-\frac{\sqrt{3}}{2}$, $\sin.3\theta=0$; unde ob signa paria casus primus praebet

$$\Sigma=l\,\frac{\sin.30^{0}(1+r)}{\sin.30^{0}(1-r)}+l\,\frac{\sin.30^{0}(2+r)}{\sin.30^{0}(2-r)},$$

quae convenire debet cum superiore pro casu, quo $\mu:\nu=2:3$.

5°. Sit $\mu=5$ ideoque $\theta=150°$, ergo $\sin.\theta=\frac{1}{2}$, $\sin.2\theta=-\frac{\sqrt{3}}{2}$, $\sin.3\theta=1$; unde ob signa disparia secundus casus nobis dat

$$\Sigma=l\,\frac{\text{tang.}\,15^{0}(1+r)}{\text{tang.}\,15^{0}(1-r)}+\sqrt{3}\cdot l\,\frac{\text{tang.}\,15^{0}(2+r)}{\text{tang.}\,15^{0}(2-r)}+2\,l\,\text{tang.}\,15^{0}(3+r).$$

EXEMPLUM 6 QUO $\nu=\infty$

35. Quia igitur fractio $\frac{\mu}{\nu}$ ut evanescens spectatur, ponamus $\mu=1$ sicque angulus θr prae πr evanescet; unde cum loco sinuum angulorum θ et θr ipsos angulos ponere liceat, erit noster valor

$$\Sigma=\pi\int\frac{r\partial r}{\sin.\pi r}.$$

Deinde quia etiam angulus $\varrho=\frac{\pi}{2\nu}$ in nihilum abit, loco omnium sinuum in expressione pro Σ inventa occurrentium ipsos angulos scribere licebit, quo observato valor quantitatis Σ sequenti modo exprimetur

$$l\,\frac{1+r}{1-r}-2\,l\,\frac{2+r}{2-r}+3\,l\,\frac{3+r}{3-r}-4\,l\,\frac{4+r}{4-r}+\text{etc.}\ ^{1)}$$

1) Haec quidem series divergens est, sed formula ab EULERO in § 37 data hoc modo demonstrari potest.

Ponendo $\mu=1$, $\nu=2n+1$, ita ut sit

$$\theta=2\varrho=\frac{\pi}{2n+1},$$

ex casu primo nanciscimur

$$\Sigma=\frac{\pi}{\sin.\theta}\int_{0}^{r}\frac{\sin.\theta r\,dr}{\sin.\pi r}=\sum_{j=1}^{j=n}(-1)^{j-1}\frac{\sin.j\theta}{\sin.\theta}\log.\frac{\sin.(j+r)\theta}{\sin.(j-r)\theta},$$

quae expressio, si ponamus

$$\frac{\text{tang.}\,r\theta}{\text{tang.}\,j\theta}=x_{j},$$

36. Singuli hi logarithmi commode in series resolvi possunt. Cum enim forma generalis omnium terminorum sit $il\frac{i+r}{i-r}$, tum vero per notam resolutionem sit

$$l\frac{i+r}{i-r}=\frac{2r}{i}+\frac{2r^3}{3i^3}+\frac{2r^5}{5i^5}+\frac{2r^7}{7i^7}+\text{etc.},$$

erit totum membrum

$$=2r\left(1+\frac{rr}{3ii}+\frac{r^4}{5i^4}+\frac{r^6}{7i^6}+\text{etc.}\right);$$

quamobrem singulis partibus hoc modo evolutis fiet

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hanc induet formam

$$\Sigma=\sum_{j=1}^{j=n}(-1)^{j-1}\frac{\sin. j\theta}{\sin.\theta}\log.\frac{1+x_j}{1-x_j}.$$

Cum autem sit $r<1$ et $n\theta<\frac{\pi}{2}$, erit x_j pro $j=1, 2, \ldots n$ numerus positivus unitate minor, quamobrem hic scribere licet

$$\log.\frac{1+x_j}{1-x_j}=2x_j\left(1+\frac{1}{3}x_j^2+\frac{1}{5}x_j^4+\frac{1}{7}x_j^6+\cdots\right).$$

Hoc modo ponendo

$$\sum_{j=1}^{j=n}(-1)^{j-1}x_j^{2i}\cos. j\theta=S_i$$

habebimus

$$\Sigma=2\frac{\text{tang.} r\theta}{\sin.\theta}\left(S_0+\frac{1}{3}S_1+\frac{1}{5}S_2+\frac{1}{7}S_3+\cdots\right),$$

ubi, cum sit

$$x_{j+1}<x_j,\quad \cos.(j+1)\theta<\cos. j\theta,\quad x_1<r,$$

erit

$$S_i>0 \quad\text{et}\quad S_i<r^{2i}.$$

Ex his perspicitur numero n ad infinitum crescente fore

$$\lim\Sigma=\pi\int_0^r\frac{r\,dr}{\sin.\pi r}=2r\left(\lim S_0+\frac{1}{3}\lim S_1+\frac{1}{5}\lim S_2+\cdots\right).$$

Quae est Euleri formula, quia pro $i>0$ manifestum est fore

$$\lim S_i=r^{2i}\left(1-\frac{1}{2^{2i}}+\frac{1}{3^{2i}}-\frac{1}{4^{2i}}+\cdots\right)$$

et ponendo $i=0$ habemus

$$S_0=\frac{1}{2}+\frac{(-1)^{n-1}\cos.\left(n+\frac{1}{2}\right)\theta}{2\cos.\frac{\theta}{2}}$$

sive, cum sit $\theta=\frac{\pi}{2n+1}$,

$$S_0=\frac{1}{2}.$$

A. L

$$\begin{aligned}\frac{\Sigma}{2r} = &+1+\frac{rr}{3}+\frac{r^4}{5}+\frac{r^6}{7}+\frac{r^8}{9}+\text{etc.}\\ &-1-\frac{rr}{3\cdot 4}-\frac{r^4}{5\cdot 4^2}-\frac{r^6}{7\cdot 4^3}-\frac{r^8}{9\cdot 4^4}-\text{etc.}\\ &+1+\frac{rr}{3\cdot 9}+\frac{r^4}{5\cdot 9^2}+\frac{r^6}{7\cdot 9^3}+\frac{r^8}{9\cdot 9^4}+\text{etc.}\\ &-1-\frac{rr}{3\cdot 16}-\frac{r^4}{5\cdot 16^2}-\frac{r^6}{7\cdot 16^3}-\frac{r^8}{9\cdot 16^4}-\text{etc.}\\ &\qquad\qquad\text{etc.}\end{aligned}$$

37. Quodsi iam istas series secundum columnas verticales disponamus, quia prima columna dat

$$1-1+1-1+1-1+\text{etc.}=\frac{1}{2},$$

prodibit haec expressio

$$\begin{aligned}\frac{\Sigma}{2r} = \frac{1}{2}&+\frac{1}{3}rr\left(1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{etc.}\right)\\ &+\frac{1}{5}r^4\left(1-\frac{1}{4^2}+\frac{1}{9^2}-\frac{1}{16^2}+\frac{1}{25^2}-\text{etc.}\right)\\ &+\frac{1}{7}r^5\left(1-\frac{1}{4^3}+\frac{1}{9^3}-\frac{1}{16^3}+\frac{1}{25^3}-\text{etc.}\right)\\ &\qquad\qquad\text{etc.}\end{aligned}$$

Quoniam igitur harum serierum omnium summae sunt cognitae, hinc per approximationem eo facilius valor litterae Σ definiri poterit, quia littera r semper denotat fractionem unitate minorem.

38. Quodsi ergo in subsidium vocemus ea, quae olim[1]) circa summas harum potestatum erueram, atque iisdem denominationibus utamur ponendo

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1) Vide EULERI Commentationes 41 et 61 (indicis ENESTROEMIANI): *De summis serierum reciprocarum,* Comment. acad. sc. Petrop. 7 (1734/5), 1740, p. 123, et *De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera: in qua eaedem summationes ex fonte maxime diverso derivantur,* Miscellanea Berolin. 7, 1743, p. 172; *LEONHARDI EULERI Opera omnia,* series I, vol. 14. Vide etiam N. BERNOULLI, *Inquisitio in summam seriei*

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}\ \text{etc.},$$

$$A\pi^2 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.},$$
$$B\pi^4 = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \text{etc.},$$
$$C\pi^6 = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \text{etc.}$$
$$\text{etc.},$$

quoniam hinc facile summae derivantur, quando terminorum signa alternantur, habebitur

$$\frac{\Sigma}{2r} = \frac{1}{2} + \frac{1}{3}\left(1 - \frac{1}{2}\right)A\pi\pi rr + \frac{1}{5}\left(1 - \frac{1}{8}\right)B\pi^4 r^4 + \frac{1}{7}\left(1 - \frac{1}{32}\right)C\pi^6 r^6 + \text{etc.}$$

Ubi meminisse convenit esse

$$A = \frac{1}{6}, \quad B = \frac{1}{90}, \quad C = \frac{1}{945}, \quad D = \frac{1}{9450}, \quad E = \frac{1}{93555} \quad \text{etc.}$$

Horum autem valorum ratio iam saepius abunde est exposita.

Comment. acad. sc. Petrop. **10** (1738), 1747, p. 19, nec non epistolas a N. BERNOULLI d. 13. Iulii et 24. Octobris 1742 ad EULERUM scriptas, *Correspondance math. et phys. publiée par P. H. Fuss*, St.-Pétersbourg 1843, t. II, p. 681 et 690; *LEONHARDI EULERI Opera omnia*, series III. Vide praeterea EULERI *Introductionem in analysin infinitorum*, Lausannae 1748, t. I cap. X; *LEONHARDI EULERI Opera omnia*, series I, vol. 8. A. L.

EVOLUTIO FORMULAE INTEGRALIS

$$\int \partial x \left(\frac{1}{1-x} + \frac{1}{lx}\right)$$

A TERMINO $x=0$ USQUE AD $x=1$ EXTENSAE

Conventui exhibita die 29. Februarii 1776

Commentatio 629 indicis Enestroemiani
Nova acta academiae scientiarum Petropolitanae 4 (1786), 1789, p. 3—16
Summarium ibidem p. 109—110

SUMMARIUM

Dans un mémoire intitulé *Observationes de progressionibus harmonicis*[1]), qui se trouve dans le 7ᵉ volume des Commentaires de l'Académie pour les années 1734 et 1735, feu M. Euler avoit déjà fait des recherches sur la somme de la progression harmonique $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{i}$, et il l'avoit trouvée $= l(i+1) + C$, i étant un nombre infiniment grand et $C = 0{,}57721$ une constante introduite par l'intégration, et qui exprime par conséquent la différence entre la somme de la série $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{\infty}$ et $l\infty$.

M. Euler a poussé plus loin cette recherche dans la suite, dans son ouvrage *Institutiones Calculi Differentialis*[2]), où l'on trouve, dans le 6ᵉ chapitre de la seconde section, qui traite *De summatione progressionum per series infinitas*, le calcul de la constante C poussé jusqu'à 16 décimales, savoir $C = 0{,}5772156649015325.$[3]) (*Inst. Calc. Diff.*)

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1) Voir la note 2 p. 319. A. L.

2) L. Euleri *Institutiones calculi differentialis*, Petropoli 1755; *Leonhardi Euleri Opera omnia*, series I, vol. 10, p. 339. A. L.

3) Voir la note 3 p. 319. A. L

Ce nombre C, trouvé par approximation, avoit paru à M. EULER assez remarquable pour l'engager à en faire le sujet d'un mémoire plus récent[1]), inséré dans le second volume des Actes de l'Académie, pour l'année 1781, où notre Géomètre a tâché de réduire ce nombre à une expression connue. Il y a donné pour cet effet plusieurs séries assez convergentes et régulières dont ce nombre C est la somme, et entre autres il y a démontré que C est la valeur de l'intégrale $\int \partial x\left(\frac{1}{1-x}+\frac{1}{lx}\right)$, prise depuis $x=0$ jusqu'à $x=1$. Or n'ayant pu réussir, dans le mémoire mentionné, à réduire ce nombre C à une quantité transcendante déjà connue, son but est d'examiner ici, si la résolution de cette formule intégrale ne mènera pas à quelque résultat plus satisfaisant.

Il essaye premièrement la voye des Quadratures, en considérant une ligne courbe dont l'abscisse est x et l'ordonnée $y=\frac{1}{1-x}+\frac{1}{lx}$. Il en examine la figure singulière, ce qui, quoiqu'il ne contribue rien à la connoissance plus parfaite du nombre C représenté par un espace déterminé de cette courbe, n'est pas pourtant destitué de tout intérêt.

L'Auteur essaye, après cet examen, plusieurs transformations en séries, qui, quoique très-instructives et neuves en partie, conduisent à des séries ni de forme connue ni assez convergentes pour qu'on pût en tirer quelque avantage pour la connoissance plus parfaite du nombre en question.

1. Ista formula integralis eo magis est notatu digna, quod eius valorem ostendi convenire cum eo, quem praebet ista expressio

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-ln,$$

si numerus n sumatur infinite magnus, et quem per approximationem olim[2]) inveni esse $=0{,}5772156649015325$[3]), cuius valorem nullo adhuc modo ad

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1) Voir la note p. 326. A. L.

2) Vide L. EULERI Commentationem 43 (indicis ENESTROEMIANI): *De progressionibus harmonicis observationes*, Commment. acad. sc. Petrop. 7 (1734/5), 1740, p. 150; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. Vide porro L. EULERI *Institutionum calculi integralis* vol. 1, Petropoli 1768, sectio 1, cap. IV; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 122. Imprimis autem vide L. MASCHERONII *Adnotationes ad calculum integralem EULERI*, Ticini 1790/2; *LEONHARDI EULERI Opera omnia*, series I, vol. 12, p. 423 et 502. A. L.

3) Secundum calculos recentiores ultima figura decimalis 5 mutanda est in 9; vide *LEONHARDI EULERI Opera omnia*, series I, vol. 11, notam p. 339, atque imprimis vol. 12, notam 2 p. 431. A. L.

mensuras transcendentes iam cognitas redigere potui; unde haud inutile erit resolutionem huius formulae propositae pluribus modis tentare. Ac primo quidem, quoniam duabus constat partibus

$$\int \frac{\partial x}{1-x} \quad \text{et} \quad \int \frac{\partial x}{lx},$$

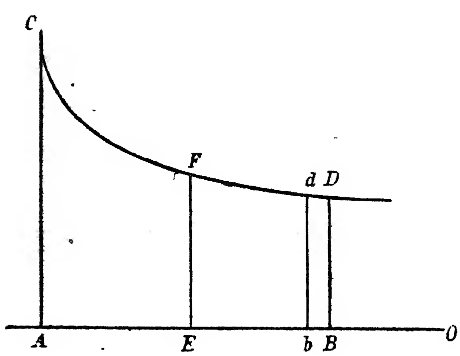
manifestum est prioris partis valorem $-l(1-x)$ posito $x=1$ fore $-l0$ ideoque $=\infty$; tum vero etiam facile perspicitur posterioris partis valorem quoque esse infinitum, sed signo contrario affectum, ita ut haud difficulter intelligatur aggregatum earum partium finitum habere valorem.

EVOLUTIO PRIMA GEOMETRICA

2. Primo igitur hanc formulam per quadraturas exhibeamus considerando lineam curvam, cuius abscissae x respondeat applicata

$$y=\frac{1}{1-x}+\frac{1}{lx};$$

tum vero eius area $\int y \partial x$ abscissae x insistens ipsum valorem quaesitum repraesentabit, quamobrem formam huius curvae accuratius perpendamus. Ac primo quidem evidens est hanc curvam neutiquam in regionem abscissarum negativarum porrigi, sed a termino $x=0$ incipere. Posito autem $x=0$ manifesto fit $y=1$ ob $lx=\infty$; at existente x infinite parvo fiet $y=1+x+\frac{1}{lx}$, ubi facile perspicitur postremum membrum $\frac{1}{lx}$ esse negativum et quasi infinities maius quam x, ita ut fiat $y=1-ix$ existente i numero maximo; unde patet, si curvam ad axem AO referamus in eoque abscissas x a puncto A capiamus, in ipso puncto A applicatam fore $AC=1$ et curvam in C hanc applicatam AC tangere, propterea quod decrementum applicatae infinities superat incrementum abscissae. Curva igitur originem ducet ab ipso puncto C hincque continuo propius ad axem inflectetur, quem tandem in distantia infinita attinget. Posito enim $x=\infty$ fit



$$y=-\frac{1}{\infty}+\frac{1}{l\infty};$$

ubi notetur prius membrum $\frac{1}{\infty}$ prae altero evanescere, ita ut iste valor sit positivus, unde patet hanc curvam a puncto C ad axem continuo propius esse accessuram.

3. Consideremus nunc abscissam $AB=1$, ubi sumto $x=1$ fit $y=\frac{1}{0}+\frac{1}{0}$, unde nihil plane concludere liceret; hanc ob causam statuamus $x=1-\omega$, ut fiat $y=\frac{1}{\omega}+\frac{1}{l(1-\omega)}$. Iam $l(1-\omega)$ in seriem evolvendo fiet

$$y=\frac{1}{\omega}-\frac{1}{\omega+\frac{1}{2}\omega^2+\frac{1}{3}\omega^3+\text{etc.}}=\frac{\frac{1}{2}+\frac{1}{3}\omega}{1+\frac{1}{2}\omega+\frac{1}{3}\omega^2}.$$

Fiat nunc $\omega=0$ ac manifestum est applicatam in puncto B fore $BD=\frac{1}{2}$, cum esset $AC=1$ et $AB=1$. Hinc simul patet sumto ω minimo, scilicet $Bb=\omega$, fore applicatam in hoc puncto

$$bd=\frac{\frac{1}{2}+\frac{1}{3}\omega}{1+\frac{1}{2}\omega}=\frac{1}{2}+\frac{1}{12}\omega$$

sicque elementum curvae Dd ad axem inclinatur sub angulo, cuius tangens est $\frac{1}{12}$, qui est propemodum $4^0 46'$.

4. Sumamus nunc abscissam $AE=\frac{1}{2}$ eique respondebit applicata $EF=2-\frac{1}{l2}=0{,}557$ propemodum atque hinc iam proxime aream $ABCD$ colligere licet. Namque si CF esset linea recta, foret area $ACFE=0{,}389$; quia autem versus axem incurvatur, haec area erit aliquanto minor. Pro altera parte, quia FD minus incurvatur, erit area $BDFE$ aliquantillum minor quam $\frac{1}{2}BE(BD+EF)=0{,}264$ propemodum, unde tota area $ACDB$ certe minor erit quam $0{,}653$, id quod iam satis convenit cum veritate, quandoquidem haec area esse debet $0{,}577$. At si abscissam $AB=1$ in plures partes dividere et areas singulis partibus respondentes indagare vellemus, earum summa eo propius ad valorem cognitum accessura foret, quo plures partes fuerint constitutae. Quia autem de vero valore huius formulae iam certi sumus, talis labor frustra susciperetur, sed hic sufficiat formam huius curvae prorsus singularis, quippe quae in puncto C subito incipit, expendisse.

EVOLUTIO SECUNDA

5. Evolvamus nunc lx in seriem, et quia est $x=1-(1-x)$, erit

$$lx=-(1-x)-\frac{1}{2}(1-x)^2-\frac{1}{3}(1-x)^3-\frac{1}{4}(1-x)^4-\text{etc.};$$

et quia est

$$y=\frac{1}{1-x}+\frac{1}{lx}=\frac{lx+1-x}{(1-x)lx},$$

istam seriem tantum in numeratore loco lx scribamus prodibitque

$$y=\frac{-\frac{1}{2}(1-x)^2-\frac{1}{3}(1-x)^3-\frac{1}{4}(1-x)^4-\text{etc.}}{(1-x)lx}$$

atque hinc

$$y=\frac{-\frac{1}{2}(1-x)-\frac{1}{3}(1-x)^2-\frac{1}{4}(1-x)^3-\text{etc.}}{lx};$$

hinc igitur per partes integrando valor quaesitus erit

$$\int y\partial x=-\frac{1}{2}\int\frac{(1-x)\partial x}{lx}-\frac{1}{3}\int\frac{(1-x)^2\partial x}{lx}-\frac{1}{4}\int\frac{(1-x)^3\partial x}{lx}-\text{etc.},$$

quae formulae singulae facile ad formulam illam generalem reducuntur, qua ostendi esse

$$\int\frac{x^m-x^n}{lx}\partial x=l\frac{m+1}{n+1}.\text{*)}$$

Hinc enim statim erit

$$\int\frac{(1-x)\partial x}{lx}=l\frac{1}{2},$$

et quia est $(1-x)^2=1-x-(x-xx)$, erit

$$\int\frac{(1-x)^2\partial x}{lx}=l\frac{1}{2}-l\frac{2}{3}=l\frac{1\cdot 3}{2^2}.$$

*) Hoc integrale duplici modo ab Ill. huius dissertationis Auctore fuit inventum in Tomo XIX Novorum Commentariorum, p. 70 et 79.[1])

1) Scilicet in Commentatione 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421. Vide etiam Commentationem 587 huius voluminis, imprimis § 9. A. L.

Simili modo facile patebit fore

$$\int\frac{(1-x)^3\partial x}{lx}=l\frac{1\cdot 3^3}{2^3\cdot 4},$$

$$\int\frac{(1-x)^4\partial x}{lx}=l\frac{1\cdot 3^6\cdot 5}{2^4\cdot 4^4},$$

$$\int\frac{(1-x)^5\partial x}{lx}=l\frac{1\cdot 3^{10}\cdot 5^5}{2^5\cdot 4^{10}\cdot 6},$$

$$\int\frac{(1-x)^6\partial x}{lx}=l\frac{1\cdot 3^{15}\cdot 5^{15}\cdot 7}{2^6\cdot 4^{20}\cdot 6^6}$$

etc.

6. Ex his igitur valor nostrae formulae $\int y\partial x$ per seriem logarithmicam prorsus singularem sequenti modo exprimetur

$$\int y\partial x=\frac{1}{2}l2+\frac{1}{3}l\frac{2^2}{1\cdot 3}+\frac{1}{4}l\frac{2^3\cdot 4}{1\cdot 3^3}+\frac{1}{5}l\frac{2^4\cdot 4^4}{1\cdot 3^6\cdot 5}+\frac{1}{6}l\frac{2^5\cdot 4^{10}\cdot 6}{1\cdot 3^{10}\cdot 5^5}$$
$$+\frac{1}{7}l\frac{2^6\cdot 4^{20}\cdot 6^6}{1\cdot 3^{15}\cdot 5^{15}\cdot 7}+\text{etc.}$$

Ubi probe notandum est omnes logarithmos capi debere hyperbolicos; facile autem intelligitur terminos huius seriei continuo prodire minores neque tamen hanc seriem tantopere convergere, ut ex ea valor quaesitus commode computari possit.

EVOLUTIO TERTIA

7. Utamur eadem resolutione logarithmi x in seriem infinitam ac ponamus brevitatis gratia $1-x=t$, ut sit

$$lx=-t-\frac{1}{2}tt-\frac{1}{3}t^3-\frac{1}{4}t^4-\text{etc.},$$

eritque

$$\frac{1}{lx}=\frac{-1}{t(1+\frac{1}{2}t+\frac{1}{3}tt+\frac{1}{4}t^3+\frac{1}{5}t^4+\text{etc.})}.$$

Iam fractionem $\frac{1}{1+\frac{1}{2}t+\frac{1}{3}tt+\text{etc.}}$ convertamus more solito in seriem recurrentem, quae sit

$$1+\alpha t+\beta tt+\gamma t^3+\delta t^4+\varepsilon t^5+\zeta t^6+\text{etc.},$$

ubi coefficientes α, β, γ, δ etc. ita erunt comparati, ut sit

$$\begin{array}{ll}
\alpha+\frac{1}{2}=0, & \text{hincque}\quad \alpha=-\frac{1}{2}, \\
\beta+\frac{1}{2}\alpha+\frac{1}{3}=0, & \beta=-\frac{1}{12}, \\
\gamma+\frac{1}{2}\beta+\frac{1}{3}\alpha+\frac{1}{4}=0, & \gamma=-\frac{1}{24}, \\
\delta+\frac{1}{2}\gamma+\frac{1}{3}\beta+\frac{1}{4}\alpha+\frac{1}{5}=0 & \delta=-\frac{19}{720} \\
\text{etc.} & \text{etc.,}
\end{array}$$

unde hanc seriem tamquam cognitam spectare licet.

8. Hoc igitur valore substituto erit

$$\frac{1}{lx}=-\frac{1}{t}-\alpha-\beta t-\gamma tt-\delta t^3-\varepsilon t^4-\text{etc.},$$

quare, cum sit $\frac{1}{1-x}=\frac{1}{t}$, erit

$$y=-\alpha-\beta t-\gamma tt-\delta t^3-\varepsilon t^4-\text{etc.}$$

sive

$$y=-\alpha-\beta(1-x)-\gamma(1-x)^2-\delta(1-x)^3-\text{etc.}$$

Cum nunc in genere sit

$$\int\partial x(1-x)^n=C-\frac{(1-x)^{n+1}}{n+1}=\frac{1}{n+1}-\frac{(1-x)^{n+1}}{n+1},$$

posito $x=1$, quemadmodum assumsimus, erit

$$\int\partial x(1-x)^n=\frac{1}{n+1}.$$

Hinc igitur singulis integralibus collectis reperietur

$$\int y\partial x=-\frac{\alpha}{1}-\frac{\beta}{2}-\frac{\gamma}{3}-\frac{\delta}{4}-\frac{\varepsilon}{5}-\text{etc.,}$$ [1]

1) **Editio princeps:**

$$\int y\partial x=-\frac{\alpha}{2}-\frac{\beta}{3}-\frac{\gamma}{4}-\frac{\delta}{5}-\frac{\varepsilon}{6}-\text{etc.},$$

unde per valores ante evolutos fiet

$$\int y\partial x=\frac{1}{4}+\frac{1}{36}+\frac{1}{96}+\frac{19}{3600}+\text{etc.}$$

Correxit A. L.

unde per valores ante evolutos fiet

$$\int y\partial x = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \text{etc.},$$

quae series utique parum est convergens.

EVOLUTIO QUARTA

9. Cum habeamus $y = \frac{lx+1-x}{(1-x)lx}$, quemadmodum ante partem lx in seriem infinitam resolvimus, ita nunc vicissim ipsam quantitatem in seriem per logarithmos ipsius x procedentem evolvamus. Quia enim est $x = e^{lx}$, erit

$$x = 1 + lx + \frac{1}{2}(lx)^2 + \frac{1}{6}(lx)^3 + \frac{1}{24}(lx)^4 + \text{etc.},$$

ubi loco lx brevitatis ergo scribamus u, atque hanc seriem tantum in numeratorem introducamus, ut fiat

$$y = \frac{-\frac{1}{2}uu - \frac{1}{6}u^3 - \frac{1}{24}u^4 - \frac{1}{120}u^5 - \text{etc.}}{u(1-x)}$$

sive

$$y = \frac{-\frac{1}{2}u - \frac{1}{6}uu - \frac{1}{24}u^3 - \frac{1}{120}u^4 - \text{etc.}}{1-x}$$

ideoque

$$\int y\partial x = -\frac{1}{2}\int\frac{\partial x lx}{1-x} - \frac{1}{6}\int\frac{\partial x(lx)^2}{1-x} - \frac{1}{24}\int\frac{\partial x(lx)^3}{1-x} - \frac{1}{120}\int\frac{\partial x(lx)^4}{1-x} - \text{etc.}$$

10. Cum nunc in genere, sumto scilicet integrali ab $x=0$ ad $x=1$, sit

$$\int \partial x(lx)^n = \pm 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n,$$

ubi signum $+$ valet, quando n est numerus par, contra vero signum $-$, erit porro

$$\int x^{n-1}\partial x(lx)^\lambda = \pm\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots\lambda}{n^{\lambda+1}},$$

ubi signum $+$ valet, si λ fuerit numerus par, inferius vero, si impar. Hinc igitur singulas nostras formulas per series integremus, dum loco $\frac{1}{1-x}$ seriem scribimus

$$1 + x + xx + x^3 + x^4 + x^5 + \text{etc.},$$

atque hinc primo nanciscemur

$$\int \frac{\partial x lx}{1-x} = -1\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}\right),$$

$$\int \frac{\partial x (lx)^2}{1-x} = 1\cdot 2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+\text{etc.}\right),$$

$$\int \frac{\partial x (lx)^3}{1-x} = -1\cdot 2\cdot 3\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}\right),$$

$$\int \frac{\partial x (lx)^4}{1-x} = 1\cdot 2\cdot 3\cdot 4\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\text{etc.}\right)$$

etc.

His igitur seriebus substitutis reperiemus

$$\begin{aligned}\int y\partial x = &\ \frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}\right)\\ &-\frac{1}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}\right)\\ &+\frac{1}{4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}\right)\\ &-\frac{1}{5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}\right)\\ &\quad \text{etc.},\end{aligned}$$

cuius expressionis iam nuper ostendi valorem esse numerum illum memorabilem 0,5772156649015325.*)

EVOLUTIO QUINTA

11. Utamur hic eadem resolutione in seriem ipsius numeri x, sed eam alio modo adhibeamus. Scilicet cum posito $lx = u$ sit

$$x = 1 + u + \frac{1}{2}uu + \frac{1}{6}u^3 + \frac{1}{24}u^4 + \frac{1}{120}u^5 + \text{etc.},$$

*) Videatur Dissertatio *De numero memorabili in summatione progressionis harmonicae naturalis occurrente.* Acta Acad. pro Anno 1781. Pars posterior, p. 49 seq.[1]) F[USS].

1) Quae dissertatio est Commentatio 583 (indicis ENESTROEMIANI), *LEONHARDI EULERI Opera omnia*, series I, vol. 15. A. L.

erit formulae nostrae ipsa pars prior

$$\frac{1}{1-x}=\frac{-1}{u+\frac{1}{2}uu+\frac{1}{6}u^3+\frac{1}{24}u^4+\frac{1}{120}u^5+\text{etc.}}=-\frac{1}{u}\cdot\frac{1}{1+\frac{1}{2}u+\frac{1}{6}uu+\frac{1}{24}u^3+\text{etc.}}.$$

Hanc fractionem

$$\frac{1}{1+\frac{1}{2}u+\frac{1}{6}uu+\frac{1}{24}u^3+\text{etc.}}$$

more solito in seriem recurrentem convertamus, quae sit

$$1-Au+Buu-Cu^3+Du^4-Eu^5+\text{etc.},$$

eritque facta comparatione

$$\begin{array}{ll}
A=\frac{1}{2}, & \text{ergo}\quad A=\frac{1}{2},\\[2mm]
B=\frac{1}{2}A-\frac{1}{6}, & B=\frac{1}{12},\\[2mm]
C=\frac{1}{2}B-\frac{1}{6}A+\frac{1}{24}, & C=0,\\[2mm]
D=\frac{1}{2}C-\frac{1}{6}B+\frac{1}{24}A-\frac{1}{120}, & D=-\frac{1}{720},\\[2mm]
E=\frac{1}{2}D-\frac{1}{6}C+\frac{1}{24}B-\frac{1}{120}A+\frac{1}{720} & E=0\\[2mm]
\qquad\text{etc.}, & \text{etc.}
\end{array}$$

12. Hac igitur serie introducta et loco u restituto valore lx formula nostra $\frac{1}{1-x}+\frac{1}{lx}=y$ sequentem induet formam

$$y=-\frac{1}{u}+A-Bu+Cuu-Du^3+\text{etc.}+\frac{1}{u}$$

sive

$$y=A-Blx+C(lx)^2-D(lx)^3+E(lx)^4-\text{etc.},$$

unde, cum in genere sit

$$\int\partial x(lx)^n=\pm 1\cdot 2\cdot 3\cdot 4\cdots n,$$

postquam scilicet absoluta integratione positum fuerit $x=1$, ubi signum

superius valet, quando n est numerus par, inferius vero, si n impar, hoc observato nanciscimur valorem quaesitum $\int y\partial x$ sequenti modo expressum

$$\int y\partial x = A + 1B + 1\cdot 2C + 1\cdot 2\cdot 3D + 1\cdot 2\cdot 3\cdot 4E + 1\cdot 2\cdots 5F + \text{etc.}^{1)},$$

quae series utique parum convergit ob coefficientes litterarum A, B, C, D etc.; verum perpendendum est ipsos valores harum litterarum continuo magis decrescere, quandoquidem certum est seriei valorem esse debere 0,5772156649015325, quare operae pretium erit harum litterarum seriem accuratius evolvere.

TRANSFORMATIO FRACTIONIS

$$\frac{1}{1+\frac{1}{2}u+\frac{1}{6}uu+\frac{1}{24}u^3+\frac{1}{120}u^4+\text{etc.}}$$

IN SERIEM

$$1 - Au + Buu - Cu^3 + Du^4 - Eu^5 + \text{etc.}$$

13. Designet littera s summam huius seriei eritque $s=\frac{u}{e^u-1}$, unde fit $e^u=\frac{u+s}{s}$ hincque $u=l(u+s)-ls$, ergo differentiando erit

$$\partial u = \frac{\partial u+\partial s}{u+s}-\frac{\partial s}{s}=\frac{s\partial u-u\partial s}{s(u+s)};$$

sive statim ponatur $s=pu$, ut sit $u=l\frac{1+p}{p}$, unde fit $\partial u=-\frac{\partial p}{p(p+1)}$; quae expressio quo seriem praebeat concinniorem, statuamus $p=q-\frac{1}{2}$, ut iam sit $s=\left(q-\frac{1}{2}\right)u$; tum vero erit $\partial u=-\frac{\partial q}{qq-\frac{1}{4}}$, unde colligitur haec aequatio

$$qq-\frac{1}{4}+\frac{\partial q}{\partial u}=0.$$

14. Ex hac igitur aequatione investigari debet series valorem ipsius q exhibens, ubi ante omnia principium huius seriei inde constitui oportet, quod

1) Haec evolutio fieri non potest, quia formula

$$\frac{u}{e^u-1}=1-Au+Bu^2-Cu^3+\cdots$$

pro $u<-2\pi$ non valet. A. L.

posito $u=0$ fieri debeat $s=1$ et $q=\frac{1}{u}+\frac{1}{2}$; unde patet seriei pro q fingendae primum terminum esse debere $\frac{1}{u}$; tum vero facile perspicitur in hac serie potestates ipsius u tantum impares assumi debere. Quamobrem fingatur ista series

$$q=\frac{1}{u}+au+bu^3+cu^5+du^7+eu^9+\text{etc.}$$

eritque

$$\begin{aligned}
qq=\frac{1}{uu}+2a+2bu^2&+\;2cu^4+\;2du^6+\;2eu^8+\;2fu^{10}+\text{etc.},\\
+aa\;&+2ab\;\;+2ac\;\;+2ad\;\;+2ae\\
&\qquad\;\;+\;bb\;\;\;+2bc\;\;+2bd\\
&\qquad\qquad\qquad\qquad\;\;+\;cc\\
\frac{\partial q}{\partial u}=-\frac{1}{uu}+\;a+3bu^2&+\;5cu^4+\;7du^6+\;9eu^8+11fu^{10}+\text{etc.};
\end{aligned}$$

harum ergo serierum summa debet esse $\frac{1}{4}$, unde deducuntur sequentes determinationes

$$\begin{aligned}
&3a=\frac{1}{4}, &&\text{ergo}\quad a=\frac{1}{12},\\
&5b+aa=0, &&b=-\frac{aa}{5},\\
&7c+2ab=0, &&c=-\frac{2ab}{7},\\
&9d+2ac+bb=0, &&d=-\frac{2ac+bb}{9},\\
&11e+2ad+2bc=0, &&e=-\frac{2ad+2bc}{11},\\
&13f+2ae+2bd+cc=0 &&f=-\frac{2ae+2bd+cc}{13}\\
&\qquad\text{etc.}, &&\qquad\text{etc.},
\end{aligned}$$

ex quibus formulis valores numerici litterarum a, b, c, d etc. computari poterunt.

15. His autem litteris a, b, c, d etc. definitis ipsa series pro s quaesita erit

$$s=1-\frac{1}{2}u+auu+bu^4+cu^6+du^8+eu^{10}+fu^{12}+gu^{14}+\text{etc.},$$

quare, cum supra posuerimus

$$s=1-Au+Buu-Cu^3+Du^4-Eu^5+Fu^6-Gu^7+Hu^8-\text{etc.},$$

valores harum litterarum maiuscularum per minusculas sequenti modo definientur

$$A=\frac{1}{2},\quad B=a,\quad C=0,\quad D=b,\quad E=0,\quad F=c,\quad G=0,\quad H=d\quad \text{etc.}$$

sicque potestatum imparium coefficientes per se evanescunt. Evidens autem est ope formularum hic inventarum valores litterarum a, b, c, d etc. multo facilius et promptius assignari posse quam per relationes supra allatas, scilicet erit

$$A=\frac{1}{2},\quad B=a=\frac{1}{12},\quad C=0,\quad D=-\frac{1}{720},\quad E=0,\quad F=\frac{1}{30240}\quad \text{etc.}$$

16. Quoniam hoc modo calculus istarum litterarum mox ad numeros vehementer magnos deduceret, loco litterarum a, b, c, d etc. quaeramus alias $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., quarum signa iam alternentur et quarum valores ad illos sequenti modo referantur

$$a=\frac{\mathfrak{A}}{12},\quad b=-\frac{\mathfrak{B}}{12^2},\quad c=+\frac{\mathfrak{C}}{12^3},\quad d=-\frac{\mathfrak{D}}{12^4},\quad e=+\frac{\mathfrak{E}}{12^5}\quad \text{etc.},$$

ita ut iam sit nostra series

$$s=1-\frac{1}{2}u+\frac{\mathfrak{A}uu}{12}-\frac{\mathfrak{B}u^4}{12^2}+\frac{\mathfrak{C}u^6}{12^3}-\frac{\mathfrak{D}u^8}{12^4}+\text{etc.},$$

atque istae novae litterae $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., sequenti modo determinabuntur

$$\mathfrak{A}=1,\quad \mathfrak{B}=\frac{\mathfrak{A}\mathfrak{A}}{5},\quad \mathfrak{C}=\frac{2\mathfrak{A}\mathfrak{B}}{7},\quad \mathfrak{D}=\frac{2\mathfrak{A}\mathfrak{C}+\mathfrak{B}\mathfrak{B}}{9},\quad \mathfrak{E}=\frac{2\mathfrak{A}\mathfrak{D}+2\mathfrak{B}\mathfrak{C}}{11},$$
$$\mathfrak{F}=\frac{2\mathfrak{A}\mathfrak{E}+2\mathfrak{B}\mathfrak{D}+\mathfrak{C}\mathfrak{C}}{13}\quad \text{etc.},$$

qui valores nunc haud difficulter in numeris evolventur ac reperientur

$$\mathfrak{A}=1,\quad \mathfrak{B}=\frac{1}{5},\quad \mathfrak{C}=\frac{2}{35},\quad \mathfrak{D}=\frac{3}{175},\quad \mathfrak{E}=\frac{2}{385},\quad \mathfrak{F}=\frac{1382}{875875}\quad \text{etc.}$$

17. Introducamus igitur istas novas litteras $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. in seriem § 12 pro $\int y\partial x$ inventam eritque

$$\int y\partial x=\frac{1}{2}+\frac{1\cdot\mathfrak{A}}{12}-\frac{1\cdot2\cdot3\mathfrak{B}}{12^2}+\frac{1\cdot2\cdot3\cdot4\cdot5\mathfrak{C}}{12^3}-\frac{1\cdot2\cdot3\cdots7\mathfrak{D}}{12^4}+\text{etc.};$$

hanc autem seriem non satis convergere iam supra observavimus.

TRANSFORMATIO FRACTIONIS

$$\frac{1}{1+\frac{1}{2}t+\frac{1}{3}tt+\frac{1}{4}t^3+\frac{1}{5}t^4+\text{etc.}}$$

IN SERIEM

$$1+\alpha t+\beta tt+\gamma t^3+\delta t^4+\varepsilon t^5+\zeta t^6+\text{etc.}$$

17[a].[1]. Ad hanc transformationem perducti sumus supra in § 7, ubi evolutio litterarum α, β, γ, δ etc. mox fiebat nimis molesta. Nunc igitur simili modo utamur quo ante positaque hac serie, quam quaerimus, $=s$ erit

$$s=\frac{-t}{l(1-t)}=tv\quad \text{ideoque}\quad l(1-t)=-\frac{1}{v},$$

ergo differentiando

$$-\frac{\partial t}{1-t}=\frac{\partial v}{vv}\quad \text{seu}\quad \frac{\partial v}{vv}+\frac{\partial t}{1-t}=0,$$

cui hanc formam tribuamus

$$vv+\frac{\partial v}{\partial t}(1-t)=0,$$

ex qua aequatione series idonea pro v elici debet.

1) In editione principe falso numerus 17 iteratur. A. L.

18. Cum igitur posito $t=0$ fiat $s=1$, hoc casu esse debebit $v=\frac{1}{t}$, quamobrem fingamus istam seriem

$$v=\frac{1}{t}+a+bt+ctt+dt^3+et^4+\text{etc.},$$

qui valor sequenti modo substituatur

$$\begin{array}{rcccccccccc}
\frac{\partial v}{\partial t}= & -\frac{1}{tt} & + * & +\,b & +\,2ct & +\,3dtt & +\,4et^3 & +\,5ft^4 & +\,6gt^5 & +\,\text{etc.}, \\
-\frac{t\partial v}{\partial t}= & & +\frac{1}{t} & -\,* & -\,bt & -\,2ctt & -\,3dt^3 & -\,4et^4 & -\,5ft^5 & -\,\text{etc.}, \\
+vv= & +\frac{1}{tt} & +\frac{2a}{t} & +\,2b & +\,2ct & +\,2dtt & +\,2et^3 & +\,2ft^4 & +\,2gt^5 & +\,\text{etc.} \\
 & & & +\,aa & +\,2ab & +\,2ac & +\,2ad & +\,2ae & +\,2af & \\
 & & & & & +\,bb & +\,2bc & +\,2bd & +\,2be & \\
 & & & & & & & +\,cc & +\,2cd &
\end{array}$$

Hinc igitur prodeunt sequentes determinationes

$$\begin{aligned}
&1+2a=0,\\
&3b+aa=0,\\
&4c+2ab-b=0,\\
&5d+2ac-2c+bb=0,\\
&6e+2ad+2bc-3d=0,\\
&7f+2ae+2bd-4e+cc=0\\
&\qquad\qquad \text{etc.},
\end{aligned}$$

quae formulae ob $a=-\frac{1}{2}$ contrahuntur in sequentes

$$3b = -\frac{1}{4}, \qquad \text{ergo} \qquad a = -\frac{1}{2},$$
$$4c = 2b, \qquad b = -\frac{1}{12},$$
$$5d = 3c - bb, \qquad c = -\frac{1}{24},$$
$$6e = 4d - 2bc, \qquad d = -\frac{19}{720},$$
$$7f = 5e - 2bd - cc, \qquad e = -\frac{3}{160},$$
$$8g = 6f - 2be - 2cd \qquad f = -\frac{863}{32\cdot 270\cdot 7}\ ^{1)}$$
$$\text{etc.}, \qquad \text{etc.}$$

19. Hinc igitur erit series quaesita

$$s = 1 + at + btt + ct^3 + dt^4 + \text{etc.},$$

quae supra [§ 7] posita fuerat

$$s = 1 + \alpha t + \beta tt + \gamma t^3 + \delta t^4 + \text{etc.};$$

litterae igitur latinae et graecae prorsus conveniunt eritque ergo

$$\int y\partial x = -\frac{a}{1} - \frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \text{etc.}$$

et valoribus modo inventis substitutis

$$\int y\partial x = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 12} + \frac{1}{3\cdot 24} + \frac{19}{4\cdot 720} + \frac{3}{5\cdot 160} + \text{etc.}\ ^{2)}$$

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1) Editio princeps: $f = -\frac{827}{32\cdot 270\cdot 7}$. Correxit A. L.

2) Editio princeps:

$$\int y\partial x = -\frac{a}{2} - \frac{b}{3} - \frac{c}{4} - \frac{d}{5} - \frac{e}{6} - \text{etc.}$$

et valoribus modo inventis substitutis

$$\int y\partial x = \frac{1}{2\cdot 2} + \frac{1}{3\cdot 12} + \frac{1}{4\cdot 24} + \frac{19}{5\cdot 720} + \frac{3}{6\cdot 160} + \text{etc.}$$

Correxit A. L.

20. Quo autem calculus harum litterarum expeditior reddatur, ponamus

$$a=-\frac{A}{2},\quad b=-\frac{B}{4},\quad c=-\frac{C}{8},\quad d=-\frac{D}{16},\quad e=-\frac{E}{32}\quad \text{etc.},$$

ut sit

$$\int y\partial x=\frac{A}{1\cdot 2}+\frac{B}{2\cdot 4}+\frac{C}{3\cdot 8}+\frac{D}{4\cdot 16}+\frac{E}{5\cdot 32}+\text{etc.};^{1)}$$

pro his autem litteris habebimus sequentes determinationes

$$A=1,\quad B=\frac{1}{3},\quad C=\frac{4B}{4},\quad D=\frac{6C+BB}{5},\quad E=\frac{8D+2BC}{6},$$

$$F=\frac{10E+2BD+CC}{7},\quad G=\frac{12F+2BE+2CD}{8}\quad \text{etc.},$$

unde colligitur

$$A=1,\quad B=\frac{1}{3},\quad C=\frac{1}{3},\quad D=\frac{19}{45},\quad E=\frac{3}{5}{}^{2)}\quad \text{etc.}$$

Haec igitur ad calculos superiores sublevandos sufficere poterunt.

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1) Editio princeps:

$$\int y\partial x=\frac{A}{2\cdot 2}+\frac{B}{3\cdot 4}+\frac{C}{4\cdot 8}+\frac{D}{5\cdot 16}+\frac{E}{6\cdot 32}+\text{etc.}$$

Correxit A. L.

2) Editio princeps: $E=\frac{39}{50}$. Correxit A. L.

UBERIOR EXPLICATIO METHODI SINGULARIS NUPER EXPOSITAE INTEGRALIA ALIAS MAXIME ABSCONDITA INVESTIGANDI

Conventui exhibita die 29. Februarii 1776

Commentatio 630 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 4 (1786), 1789, p. 17—54
Summarium ibidem p. 110—111

SUMMARIUM

Il n'y a pas longtems que le calcul intégral a été enrichi de la méthode de traiter les formules différentielles qui sont divisées par le logarithme de la quantité variable. Feu M. EULER, à qui cette branche des sciences mathématiques est redevable de ses plus brillans progrès, dit dans un de ses mémoires: *Cum mihi saepius occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant divisae, veluti* $\frac{Pdz}{lz}$, *nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda.* (V. Nov. Comment. Tom. XIX p. 66.)[1]). Cependant il s'est frayé un nouveau chemin dans le même mémoire intitulé: *Nova methodus quantitates integrales determinandi,* et il y a donné l'intégrale de plusieurs pareilles formules, et entre autres de celle-ci $\int \frac{x^a - x^b}{lx}\partial x$, dont il a trouvé, par une méthode tout à fait nouvelle, l'intégrale, prise depuis $x = 0$ jusqu'à $x = 1$, égale à $l\frac{a+1}{b+1}$.

Cette nouvelle méthode de déterminer les intégrales de beaucoup de formules qui se refusent à toute autre voye d'intégration, a paru à M. EULER susceptible de bien plus de généralisation, et propre, par là même, à avancer les bornes du calcul intégral. C'est le but de ce mémoire, qui mérite à tous égards l'attention des Géomètres, mais qui n'est pas

1) Voir la note p. 322 de ce volume et principalement p. 424 du volume précédent. A. L.

susceptible d'extrait, étant tout hérissé de calculs, et de calculs inintelligibles, sans une explication préalable des nouveaux caractères que l'Auteur s'est vu obligé d'introduire, afin de rendre ses recherches plus courtes et plus lumineuses. Explication que les connoisseurs aimeront mieux chercher, ainsi que l'esprit de la méthode, dans le mémoire même, où nous les renvoyons.

Methodus illa singularis, qua non ita pridem deductus sum ad integrationem formulae

$$\int \frac{x^a - x^b}{lx} \partial x,$$

cuius valorem a termino $x = 0$ usque ad $x = 1$ extensum inveni esse

$$l\frac{a+1}{b+1}\text{*)},$$

multo latius patet ideoque accuratiorem evolutionem meretur, quandoquidem multo maiora incrementa scientiae analyticae polliceri videtur. Quo autem hoc feliciori successu et sine ambagibus praestari possit, necesse erit peculiarem signandi modum usurpare, quem ergo ante omnia explicari conveniet.

EXPLICATIO CHARACTERUM IN SEQUENTIBUS ADHIBENDORUM

I. Si V denotet functionem quamcunque binarum variabilium x et p, tum iste character

$$\frac{\partial^\lambda}{x} \cdot V$$

mihi designabit eam quantitatem, quae oritur, si functio V solam x pro variabili sumendo toties successive differentietur, quot unitates in indice λ

*) Iam in dissertatione praecedente annotavimus Ill. Auctorem hanc integrationem exposuisse in Novorum Commentariorum Tomo XIX. pag. 70.[1])

1) Vide notam p. 322. A. L.

continentur, simulque ubique differentiale ∂x reiiciatur. Eodem modo iste character

$$\frac{\partial^\lambda}{p}\cdot V$$

designabit eam quantitatem, quae per totidem differentiationes resultat, dum sola p ut variabilis tractatur. Hinc igitur ista signandi ratio sequenti modo ad formulas usu receptas reducetur

$$\frac{\partial}{x}\cdot V=\left(\frac{\partial V}{\partial x}\right) \quad \text{et} \quad \frac{\partial}{p}\cdot V=\left(\frac{\partial V}{\partial p}\right),$$

$$\frac{\partial^2}{x}\cdot V=\left(\frac{\partial\partial V}{\partial x^2}\right) \quad \text{et} \quad \frac{\partial^2}{p}\cdot V=\left(\frac{\partial\partial V}{\partial p^2}\right),$$

$$\frac{\partial^3}{x}\cdot V=\left(\frac{\partial^3 V}{\partial x^3}\right) \quad \text{et} \quad \frac{\partial^3}{p}\cdot V=\left(\frac{\partial^3 V}{\partial p^3}\right).$$

II. Vicissim autem integrando iste character

$$\frac{\int^\lambda}{x}\cdot V$$

designabit eam quantitatem, quae ex continua integratione λ vicibus repetita oritur, dum sola x variabilis accipitur; et pariter hic character

$$\frac{\int^\lambda}{p}\cdot V$$

eam quantitatem significat, quae oritur per continuam integrationem λ vicibus repetitam, dum sola p variabilis accipitur. Haec ergo sequenti modo ad formas usu receptas revocabuntur

$$\frac{\int}{x}\cdot V=\int V\partial x \qquad \text{et} \quad \frac{\int}{p}\cdot V=\int V\partial p,$$

$$\frac{\int^2}{x}\cdot V=\int\partial x\int V\partial x \qquad \text{et} \quad \frac{\int^2}{p}\cdot V=\int\partial p\int V\partial p,$$

$$\frac{\int^3}{x}\cdot V=\int\partial x\int\partial x\int V\partial x \quad \text{et} \quad \frac{\int^3}{p}\cdot V=\int\partial p\int\partial p\int V\partial p.$$

III. At quoniam omnes quantitates per integrationem inventae per se sunt indeterminatae, in posterum perpetuo omnia integralia ita capi statuamus, ut evanescant posito vel $x=0$ vel $p=0$; prius scilicet, si sola x ut variabilis fuerit tractata, posterius vero, si sola p fuerit variabilis.

IV. Hos iam characteres pro lubitu inter se coniungere licet ac primo quidem haec formula

$$\frac{\partial^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V$$

denotat functionem V primo μ vicibus differentiari debere sumta sola x variabili, tum vero quantitatem hinc oriundam denuo ν vicibus differentiari debere sumta sola p variabili. Hinc istos characteres ad morem solitum revocando erit

$$\frac{\partial}{x}\cdot\frac{\partial}{p}\cdot V=\left(\frac{\partial\partial V}{\partial x\partial p}\right),\qquad \frac{\partial}{p}\cdot\frac{\partial}{x}\cdot V=\left(\frac{\partial\partial V}{\partial p\partial x}\right),$$

$$\frac{\partial^2}{x}\cdot\frac{\partial}{p}\cdot V=\left(\frac{\partial^3 V}{\partial x^2\partial p}\right),\qquad \frac{\partial}{x}\cdot\frac{\partial^2}{p}\cdot V=\left(\frac{\partial^3 V}{\partial x\partial p^2}\right),$$

$$\frac{\partial^3}{x}\cdot\frac{\partial^2}{p}\cdot V=\left(\frac{\partial^5 V}{\partial x^3\partial p^2}\right)\qquad \frac{\partial^2}{x}\cdot\frac{\partial^3}{p}\cdot V=\left(\frac{\partial^5 V}{\partial x^2\partial p^3}\right)$$

etc. etc.

V. Ita formula

$$\frac{\partial^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V$$

denotat functionem V primo μ vicibus differentiari debere sumta sola x variabili, tum vero quantitatem hinc oriundam ν vicibus integrari debere sumta sola p variabili. Ita si fuerit $\mu=2$ et $\nu=1$, erit more solito

$$\frac{\partial^2}{x}\cdot\frac{\int^1}{p}\cdot V=\int\partial p\left(\frac{\partial\partial V}{\partial x^2}\right),$$

unde significatio aliorum huiusmodi characterum iam satis intelligi potest.

VI. Simili modo formula hoc charactere designata

$$\frac{\int^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V$$

declarat functionem V primo μ vicibus integrari debere sumta sola x variabili, tum vero quantitatem hinc oriundam ν vicibus differentiari debere sumta sola p variabili. Quae ergo significatio satis clare perspicitur, etsi more solito non tam commode indicari posset. Si enim esset $\mu = 2$ et $\nu = 2$, valor huius formulae $\frac{\int^2}{x}\cdot\frac{\partial^2}{p}\cdot V$ hoc modo repraesentari deberet $\left(\frac{\partial\partial\int\partial x\int V\partial x}{\partial p^2}\right)$.

VII. Denique iste character

$$\frac{\int^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V$$

significat functionem V primo μ vicibus integrari debere sumta sola x pro variabili, tum vero quantitatem resultantem denuo ν vicibus integrari debere sumta sola p variabili. Ubi, quod in perpetuum est tenendum, priora integralia ita capi debent, ut evanescant posito $x = 0$, posteriora vero posito $p = 0$.

Hac characterum explicatione praemissa sequentia theoremata probe notentur, quorum veritas ex iis, quae de indole functionum duarum variabilium sunt exposita, satis clare perspicitur.

THEOREMA 1

Si V fuerit functio quaecunque duarum variabilium x et p, sequens aequalitas semper locum habebit

$$\frac{\partial^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V = \frac{\partial^\nu}{p}\cdot\frac{\partial^\mu}{x}\cdot V.$$

Hinc ergo si ponamus

$$\frac{\partial^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\partial^\nu}{p}\cdot V = R,$$

tum erit

$$\frac{\partial^\nu}{p}\cdot Q = \frac{\partial^\mu}{x}\cdot R.$$

THEOREMA 2

Si V fuerit functio quaecunque binarum variabilium x et p, tum sequens aequalitas semper locum habebit

$$\frac{\int^\mu}{x}\cdot\frac{\partial^\nu}{p}\cdot V = \frac{\partial^\nu}{p}\cdot\frac{\int^\mu}{x}\cdot V.$$

Hinc si ponamus

$$\frac{\int^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\partial^\nu}{p}\cdot V = R,$$

erit

$$\frac{\partial^\nu}{p}\cdot Q = \frac{\int^\mu}{x}\cdot R.$$

THEOREMA 3

Si fuerit V functio quaecunque binarum variabilium x et p, tum sequens aequalitas semper locum habebit

$$\frac{\partial^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V = \frac{\int^\nu}{p}\cdot\frac{\partial^\mu}{x}\cdot V.$$

Hinc si ponamus

$$\frac{\partial^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\int^\nu}{p}\cdot V = R,$$

erit

$$\frac{\int^\nu}{p}\cdot Q = \frac{\partial^\mu}{x}\cdot R.$$

THEOREMA 4

Si fuerit V functio quaecunque binarum variabilium x et p, tum sequens aequalitas semper locum habebit

$$\frac{\int^\mu}{x}\cdot\frac{\int^\nu}{p}\cdot V = \frac{\int^\nu}{p}\cdot\frac{\int^\mu}{x}\cdot V.$$

Hinc si ponamus

$$\frac{\int^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\int^\nu}{p}\cdot V = R,$$

erit

$$\frac{\int^\nu}{p}\cdot Q = \frac{\int^\mu}{x}\cdot R.$$

SCHOLION

Hae aequalitates per se ita sunt manifestae, ut quovis casu evolutae evadant identicae. Ita si sumatur

$$V = x^m p^n,$$

ex theoremate primo sumto $\mu = 2$ et $\nu = 1$ reperietur

$$Q = \frac{\partial^2}{x} \cdot V = m(m-1)x^{m-2}p^n \quad \text{et} \quad R = \frac{\partial}{p} \cdot V = np^{n-1}x^m.$$

Hinc vero elicitur

$$\frac{\partial}{p} \cdot Q = mn(m-1)x^{m-2}p^{n-1} \quad \text{et} \quad \frac{\partial^2}{x} \cdot R = mn(m-1)x^{m-2}p^{n-1},$$

qui duo valores manifesto congruunt. Ex secundo autem theoremate [sumto] $\mu = 2$ et $\nu = 1$ fiet

$$Q = \frac{\int^2}{x} \cdot V = \frac{p^n x^{m+2}}{(m+1)(m+2)} \quad \text{et} \quad R = \frac{\partial}{p} \cdot V = np^{n-1}x^m.$$

Hinc ergo erit

$$\frac{\partial}{p} \cdot Q = \frac{np^{n-1}x^{m+2}}{(m+1)(m+2)} \quad \text{et} \quad \frac{\int^2}{x} \cdot R = \frac{np^{n-1}x^{m+2}}{(m+1)(m+2)}.$$

Ex tertio theoremate manente $\mu = 2$ et $\nu = 1$ erit

$$Q = \frac{\partial^2}{x} \cdot V = m(m-1)x^{m-2}p^n \quad \text{et} \quad R = \frac{\int}{p} \cdot V = \frac{p^{n+1}x^m}{n+1}.$$

Hinc igitur erit

$$\frac{\int}{p} \cdot Q = \frac{m(m-1)x^{m-2}p^{n+1}}{n+1} \quad \text{et} \quad \frac{\partial^2}{x} \cdot R = \frac{m(m-1)x^{m-2}p^{n+1}}{n+1}.$$

Ex quarto denique theoremate erit

$$Q = \frac{\int^2}{x} \cdot V = \frac{x^{m+2}p^n}{(m+1)(m+2)} \quad \text{et} \quad R = \frac{\int}{p} \cdot V = \frac{x^m p^{n+1}}{n+1}.$$

Hinc ergo colligitur

$$\frac{\int}{p} \cdot Q = \frac{x^{m+2}p^{n+1}}{(n+1)(m+1)(m+2)} \quad \text{et} \quad \frac{\int^2}{x} \cdot R = \frac{x^{m+2}p^{n+1}}{(n+1)(m+1)(m+2)}.$$

Ob has igitur aequalitates adeo identicas nullae conclusiones hinc deduci posse videbuntur. Verum longe aliter se res habere deprehenditur, si post omnes operationes institutas ipsi x determinatus valor, veluti $x = 1$, tribui debeat, quemadmodum in quatuor problematibus sequentibus ostendemus, quae se ad quatuor theoremata praecedentia referunt.

PROBLEMA 1

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate primo indicatae absolvantur, tum vero statuatur $x=1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro primo theoremate posuimus $\frac{\partial^\mu}{x}\cdot V=Q$, deinde vero haec quantitas sola p variabili sumta differentiari debet, ita ut iam x pro constanti habeatur, statim loco x unitas scribi poterit, quo facto abeat Q in M, ita ut nunc M futura sit functio solius p. Manente igitur $R=\frac{\partial^\nu}{p}\cdot V$ consequemur hanc aequationem

$$\frac{\partial^\nu}{p}\cdot M=\frac{\partial^\mu}{x}\cdot R,$$

ubi plerumque eveniet, ut quantitas M multo promptius differentiari queat quam functio Q, unde aequalitas inventa plerumque non adeo erit obvia; id quod sequentibus exemplis illustrasse iuvabit, in quibus omnibus assumemus $V=x^{n+p}$, ita ut eius valor posito $x=1$ abeat in 1.

EXEMPLUM 1 QUO $\mu=1$ ET $\nu=1$

Hic ergo erit

$$Q=\frac{\partial}{x}\cdot x^{n+p}=(n+p)x^{n+p-1},$$

unde ergo posito $x=1$ fit

$$M=n+p;$$

quare cum sit

$$R=\frac{\partial}{p}\cdot x^{n+p}=x^{n+p}lx,$$

nanciscimur hanc aequationem $1=\frac{\partial}{x}\cdot x^{n+p}lx$. Unde patet, si post differentiationem ponatur $x=1$, fore more exprimendi solito

$$\frac{1}{\partial x}\partial.\,x^{n+p}lx=1,$$

id quod non amplius tam est obvium; est enim

$$\partial . x^{n+p} lx = (n+p) x^{n+p-1} \partial x lx + x^{n+p-1} \partial x,$$

quae expressio per ∂x divisa positoque $x=1$ abit in 1.

EXEMPLUM 2 QUO $\mu = 2$ ET $\nu = 1$

Hic igitur erit

$$Q = \frac{\partial^2}{x} \cdot x^{n+p} = (n+p)(n+p-1) x^{n+p-2};$$

posito ergo $x=1$ erit

$$M = (n+p)(n+p-1).$$

Quare cum sit

$$R = x^{n+p} lx,$$

erit

$$\frac{\partial}{p} \cdot (n+p)(n+p-1) = \frac{\partial^2}{x} \cdot x^{n+p} lx,$$

quamobrem per solitum exprimendi modum habebimus

$$\frac{\partial \partial . x^{n+p} lx}{\partial x^2} = 2(n+p) - 1,$$

postquam scilicet gemina differentiatione absoluta ponitur $x=1$.

EXEMPLUM 3 QUO $\mu = 1$ ET $\nu = 2$

Hic igitur erit

$$Q = \frac{\partial}{x} \cdot x^{n+p} = (n+p) x^{n+p-1},$$

unde posito $x=1$ fit

$$M = n+p.$$

Quare cum sit

$$R = \frac{\partial^2}{p} \cdot x^{n+p} = x^{n+p} (lx)^2,$$

erit

$$\frac{\partial^2}{p} \cdot (n+p) = \frac{\partial}{x} \cdot x^{n+p} (lx)^2$$

sive solito exprimendi more

$$\frac{\partial . x^{n+p} (lx)^2}{\partial x} = 0,$$

postquam scilicet differentiatione absoluta ponitur $x=1$.

EXEMPLUM 4 QUO $\mu=2$ ET $\nu=2$

Cum igitur hoc casu sit

$$Q=\frac{\partial^2}{x}\cdot x^{n+p}=(n+p)(n+p-1)x^{n+p-2}$$

ideoque

$$M=(n+p)(n+p-1)$$

et

$$R=\frac{\partial^2}{p}\cdot x^{n+p}=x^{n+p}(lx)^2,$$

erit

$$\frac{\partial\partial.x^{n+p}(lx)^2}{\partial x^2}=\frac{\partial\partial M}{\partial p^2}=2.$$

COROLLARIUM

Ex his exemplis iam abunde fit perspicuum, si exponentes μ et ν fuerint quicunque, tum posito $x=1$ fore

$$M=(n+p)(n+p-1)\cdots(n+p-\mu+1)$$

ideoque functionem ipsius p tantum. Quare cum sit $R=x^{n+p}(lx)^\nu$, erit more solito

$$\frac{\partial^\mu.x^{n+p}(lx)^\nu}{\partial x^\mu}=\frac{\partial^\nu M}{\partial p^\nu},$$

quando scilicet omnibus operationibus peractis statuitur $x=1$.

SCHOLION

Quemadmodum hic assumsimus $V=x^{n+p}$, ita eadem opera expedire licet hanc formam latius patentem

$$V=x^pX$$

denotante X functionem quamcunque ipsius x tantum, ita ut altera quantitas p non ingrediatur. Ponamus igitur sumto $x=1$ fieri $X=A$, $\frac{\partial X}{\partial x}=A'$, $\frac{\partial\partial X}{\partial x^2}=A''$ etc., atque cum fiat

$$Q=\frac{\partial}{x}\cdot V=px^{p-1}X+x^p\frac{\partial X}{\partial x},$$

erit hoc casu

$$M=pA+A'.$$

Deinde vero habebimus

$$\frac{\partial^2}{x}\cdot V = p(p-1)x^{p-2}X + 2px^{p-1}\frac{\partial X}{\partial x} + x^p\frac{\partial\partial X}{\partial x^2} = Q;$$

hinc ergo colligitur

$$M = p(p-1)A + 2pA' + A''.$$

Prodit porro

$$\frac{\partial^3}{x}\cdot V = p(p-1)(p-2)x^{p-3}X + 3p(p-1)x^{p-2}\frac{\partial X}{\partial x} + 3px^{p-1}\frac{\partial\partial X}{\partial x^2} + x^p\frac{\partial^3 X}{\partial x^3};$$

hinc ergo erit

$$M = p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A'''.$$

Hinc iam patet ex formula $\frac{\partial^4}{x}\cdot V$ oriturum esse valorem

$$M = p(p-1)(p-2)(p-3)A + 4p(p-1)(p-2)A' \\ + 6p(p-1)A'' + 4pA''' + A'''',$$

unde lex progressionis satis est manifesta. At vero pro altera littera R habebimus

casu $\nu = 1$ $\quad R = x^p X l x$,

casu $\nu = 2$ $\quad R = x^p X(lx)^2$,

casu $\nu = 3$ $\quad R = x^p X(lx)^3$

atque adeo in genere casu $\nu = \nu$ erit

$$R = x^p X(lx)^\nu.$$

Ex his igitur formulis nanciscemur valores differentialium omnium ordinum formulae $x^p X(lx)^\nu$, postquam factis omnibus operationibus positum fuerit $x = 1$:

$$1.\quad \frac{1}{\partial x}\partial.\, x^p X(lx)^\nu = \frac{\partial^\nu(pA + A')}{\partial p^\nu},$$

qui valor semper erit $= 0$ excepto casu $\nu = 1$, quo prodit $= A$;

$$2.\quad \frac{1}{\partial x^2}\partial\partial.\, x^p X(lx)^\nu = \frac{\partial^\nu(p(p-1)A + 2pA' + A'')}{\partial p^\nu},$$

qui valor semper est 0, quando $\nu > 2$;

$$3.\quad \frac{1}{\partial x^3}\partial^3 . x^p X(lx)^\nu = \frac{\partial^\nu(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^\nu},$$

qui valor semper evanescit exceptis casibus, quibus $\nu < 4$.

In his formulis notasse iuvabit esse:

Pro prima

$$\frac{\partial(pA + A')}{\partial p} = A;$$

pro secunda

$$\frac{\partial(p(p-1)A + 2pA' + A'')}{\partial p} = (2p-1)A + 2A'$$

et

$$\frac{\partial\partial(p(p-1)A + 2pA' + A'')}{p^2} = 2A;$$

sequentia autem sunt 0;

pro tertia

$$\frac{\partial(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p} = (3pp - 6p + 2)A + 3(2p-1)A' + 3A'',$$

$$\frac{\partial\partial(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^2} = (6p-6)A + 6A'$$

et

$$\frac{\partial^3(p(p-1)(p-2)A + 3p(p-1)A' + 3pA'' + A''')}{\partial p^3} = 6A;$$

sequentia omnia evanescunt.

PROBLEMA 2

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate secundo indicatae absolvantur, tum vero statuatur $x = 1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro secundo theoremate posuimus $\frac{\int^\mu}{x}\cdot V = Q$, deinde vero haec quantitas sumta sola p variabili ν vicibus differentiari debet posita x constante, iam ante has differentiationes ponere licet $x = 1$. Hoc

ergo facto abeat Q in M sicque habebitur $\frac{\partial^\nu}{p}\cdot Q=\frac{\partial^\nu M}{\partial p^\nu}$, quod iam est membrum primum aequalitatis quaesitae more solito expressum, quandoquidem M est sola functio ipsius p. Pro altero membro cum sit $R=\frac{\partial^\nu}{p}\cdot V$, erit hoc alterum membrum $\frac{\int^\mu}{x}\cdot R$. Quamobrem si post omnes has μ integrationes peractas (quae autem singula integralia semper ita sunt capienda, ut evanescant posito $x=0$) statuatur $x=1$, semper erit

$$\frac{\int^\mu}{x}\cdot R=\frac{\partial^\nu M}{\partial p^\nu},$$

de quo valore certi sumus, etiamsi forte integratio absolvi nequeat, quamobrem hanc veritatem exemplis illustremus, in quibus assumemus $V=x^{n+p}$.

EXEMPLUM 1 QUO $\mu=1$ ET $\nu=1$

Hoc ergo casu erit

$$Q=\int x^{n+p}\partial x=\frac{x^{n+p+1}}{n+p+1},$$

unde fit

$$M=\frac{1}{n+p+1}.$$

Deinde vero erit

$$R=\frac{\partial}{p}\cdot x^{n+p}=x^{n+p}lx,$$

ex quibus aequatio nostra fiet

$$\int x^{n+p}\partial x lx=\frac{\partial}{p}\cdot\frac{1}{n+p+1}=\frac{-1}{(n+p+1)^2}.$$

EXEMPLUM 2 QUO $\mu=1$ ET $\nu=2$

Hoc ergo casu erit

$$Q=\frac{\int}{x}\cdot x^{n+p}=\frac{x^{n+p+1}}{n+p+1}$$

ideoque M ut ante $\frac{1}{n+p+1}$. Deinde vero erit

$$R=\frac{\partial^2}{p}\cdot x^{n+p}=x^{n+p}(lx)^2,$$

quocirca posito $x=1$ habebitur ista aequatio

$$\int x^{n+p}\partial x(lx)^2 = \frac{\partial^2}{p}\cdot\frac{1}{n+p+1} = \frac{+2}{(n+p+1)^3}.$$

EXEMPLUM 3 QUO $\mu=1$ ET $\nu=3$

Hoc igitur casu erit

$$Q=\frac{x^{n+p+1}}{n+p+1} \quad \text{et} \quad M=\frac{1}{n+p+1}.$$

Tum vero erit $R=x^{n+p}(lx)^3$, unde nascitur haec aequalitas

$$\int x^{n+p}\partial x(lx)^3 = \frac{\partial^3}{p}\cdot\frac{1}{n+p+1} = \frac{-6}{(n+p+1)^4}.$$

EXEMPLUM 4 QUO $\mu=1$ ET $\nu=\nu$

Hic ex praecedentibus satis liquet aequationem hinc resultantem fore

$$\int x^{n+p}\partial x(lx)^\nu = \pm\frac{1\cdot2\cdot3\cdots\nu}{(n+p+1)^{\nu+1}},$$

ubi signum superius valet, si ν sit numerus par, inferius vero, si impar; quae reductio eo magis est notatu digna, quod alias per plures ambages ad eam perveniri solet.

EXEMPLUM 5 QUO $\mu=2$ ET $\nu=1$

Hoc ergo casu erit

$$Q=\frac{\int^2}{x}\cdot x^{n+p} = \frac{x^{n+p+2}}{(n+p+1)(n+p+2)},$$

quamobrem habebitur

$$M=\frac{1}{(n+p+1)(n+p+2)},$$

qui valor reducitur ad hunc

$$M=\frac{1}{n+p+1}-\frac{1}{n+p+2};$$

tum vero erit $R=x^{n+p}lx$, unde sequens aequalitas deducitur

$$\int\partial x\int x^{n+p}\partial x lx = \frac{-1}{(n+p+1)^2}+\frac{1}{(n+p+2)^2},$$

quae aequalitas more solito indagata iam satis molestos calculos postulat.

EXEMPLUM 6 QUO $\mu=2$ ET $\nu=2$

Hic ergo erit ut ante

$$M=\frac{1}{(n+p+1)(n+p+2)}=\frac{1}{n+p+1}-\frac{1}{n+p+2} \quad \text{et} \quad R=x^{n+p}(lx)^2$$

uti in Exemplo 2, unde statim colligitur ista aequatio

$$\int\partial x\int x^{n+p}\partial x\,(lx)^2=\frac{\partial\partial M}{\partial p^2}=\frac{2}{(n+p+1)^3}-\frac{2}{(n+p+2)^3}.$$

EXEMPLUM 7 QUO $\mu=2$ ET $\nu=\nu$

Hic ergo erit

$$M=\frac{1}{n+p+1}-\frac{1}{n+p+2} \quad \text{et} \quad R=x^{n+p}(lx)^\nu,$$

unde resultat aequatio

$$\int\partial x\int x^{n+p}\,\partial x\,(lx)^\nu=\pm\frac{1\cdot 2\cdots\nu}{(n+p+1)^{\nu+1}}\mp\frac{1\cdot 2\cdots\nu}{(n+p+2)^{\nu+1}},$$

ubi iterum signa superiora valent, si ν numerus par, inferiora vero, si impar.

EXEMPLUM 8 QUO $\mu=3$ ET $\nu=\nu$

Pro hoc casu ob

$$Q=\frac{\int^3}{x}\cdot x^{n+p}=\frac{x^{n+p+3}}{(n+p+1)(n+p+2)(n+p+3)}$$

posito $x=1$ fit

$$M=\frac{1}{(n+p+1)(n+p+2)(n+p+3)},$$

quae fractio resolvatur in suas simplices, fietque

$$M=\frac{1}{2(n+p+1)}-\frac{1}{n+p+2}+\frac{1}{2(n+p+3)},$$

unde facile patet sequentem prodituram esse aequalitatem

$$\int\partial x\int\partial x\int x^{n+p}\partial x(lx)^\nu=\pm\frac{3\cdot 4\cdots\nu}{(n+p+1)^{\nu+1}}\mp\frac{2\cdot 3\cdots\nu}{(n+p+2)^{\nu+1}}\pm\frac{3\cdot 4\cdots\nu}{(n+p+3)^{\nu+1}}$$

$$=\pm 3\cdot 4\cdot 5\cdots\nu\left(\frac{1}{(n+p+1)^{\nu+1}}-\frac{2}{(n+p+2)^{\nu+1}}+\frac{1}{(n+p+3)^{\nu+1}}\right),$$

ubi ratio signi ambigui est eadem ut ante. Facile autem intelligitur, si quis formulam illam integralem evolvere voluerit, eum in calculos valde molestos esse delapsurum.

SCHOLION

Superfluum foret indici μ maiores valores tribuere, siquidem evolutio simili modo expediri posset. Praecipuum autem negotium consistit in resolutione fractionis M in suas fractiones simplices, id quod necesse est, ut deinceps facilius omnes differentiationes atque adeo secundum indicem indefinitum ν institui queant. Hic autem labor subsidio sequentis propositionis promptissime absolvi poterit.

PROPOSITIO

Si X fuerit functio quaecunque ipsius x ac post integrationes statui debeat $x = 1$, tum semper ista formula integralis complicata

$$\frac{\int^{\mu}}{x} \cdot X$$

reduci potest ad istam formulam integralem simplicem more solito expressam

$$\frac{\int X \partial x (1-x)^{\mu-1}}{1 \cdot 2 \cdot 3 \cdots (\mu-1)}.$$

Hinc enim statim patet pro nostro casu, quo $X = x^{n+p}$, quantitatem M sequenti modo expressum iri

$$M = \frac{1}{1 \cdot 2 \cdots (\mu-1)} \left(\frac{1}{n+p+1} - \frac{\mu-1}{n+p+2} + \frac{(\mu-1)(\mu-2)}{1 \cdot 2 (n+p+3)} - \frac{(\mu-1)(\mu-2)(\mu-3)}{1 \cdot 2 \cdot 3 (n+p+4)} + \text{etc.} \right),$$

unde iam facile differentialia omnium ordinum ipsius M derivari possunt. Ceterum hic adhuc observasse iuvabit loco functionis illius V vix alium valorem accipi posse praeter x^{n+p}, propterea quod hoc solo casu omnia $\frac{\int^{\mu}}{x} \cdot V$ actu expedire licet, id quod ad nostrum institutum imprimis requiritur, quia alioquin nullae aequationes memorabiles inde deduci possent.

PROBLEMA 3

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate tertio indicatae actu absolvantur, tum vero statuatur $x=1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro tertio theoremate posuimus

$$\frac{\partial^\mu}{x}\cdot V = Q \quad \text{et} \quad \frac{\int^\nu}{p}\cdot V = R,$$

hinc deduximus sequentem aequalitatem $\frac{\int^\nu}{p}\cdot Q = \frac{\partial^\mu}{x}\cdot R$, ubi in valore pro Q invento loco x unitas scribi debet, unde resultet quantitas M, quae iam tantum erit functio ipsius p, ita ut nunc aequalitas nostra evadat

$$\frac{\partial^\nu}{p}\cdot M = \frac{\partial^\mu}{x}\cdot R.$$

Quodsi iam loco V hanc accipiamus functionem x^{n+p}, pro variis valoribus indicis μ littera M sequentes sortietur valores:

1. Si $\mu=1$, erit $M=n+p$,
2. si $\mu=2$, erit $M=(n+p)(n+p-1)$,
3. si $\mu=3$, erit $M=(n+p)(n+p-1)(n+p-2)$

etc.

hincque in genere

$$M=(n+p)(n+p-1)\cdots(n+p-\mu+1).$$

Pro littera autem R ex valoribus simplicioribus indicis ν colligetur:

$$1.\ \text{Si } \nu=1, \text{ valor } R=\frac{x^{n+p}}{lx}+C;$$

quae constans C cum ita debeat accipi, ut integrale evanescat posito $p=0$, erit hac correctione adhibita

$$R=\frac{x^{n+p}}{lx}-\frac{x^n}{lx};$$

quae formula ducta in ∂p et denuo integrata adiectaque debita constante praebet,

2. si $\nu = 2$, valorem $R = \dfrac{x^{n+p} - x^n}{(lx)^2} - \dfrac{px^n}{lx}$,

3. si $\nu = 3$, " $R = \dfrac{x^{n+p} - x^n}{(lx)^3} - \dfrac{px^n}{(lx)^2} - \dfrac{ppx^n}{2lx}$,

4. si $\nu = 4$, " $R = \dfrac{x^{n+p} - x^n}{(lx)^4} - \dfrac{px^n}{(lx)^3} - \dfrac{ppx^n}{2(lx)^2} - \dfrac{p^3 x^n}{6lx}$,

unde concluditur in genere esse proditurum

$$R = \frac{x^{n+p}}{(lx)^\nu} - x^n\left(\frac{1}{(lx)^\nu} + \frac{p}{(lx)^{\nu-1}} + \frac{pp}{1\cdot 2(lx)^{\nu-2}} + \cdots + \frac{p^{\nu-1}}{1\cdot 2\cdot 3\cdots(\nu-1)lx}\right).$$

His igitur valoribus evolutis sequentia exempla evolvamus.

EXEMPLUM 1 QUO $\mu = 1$ ET $\nu = 1$

Hoc ergo casu erit

$$M = n + p \quad \text{et} \quad R = \frac{x^{n+p} - x^n}{lx},$$

unde oritur haec aequalitas

$$\frac{1}{\partial x}\cdot\frac{\partial(x^{n+p} - x^n)}{lx} = \frac{\int}{p}\cdot(n+p) = np + \frac{pp}{2}$$

more solito expressa. Hic scilicet forma $\frac{x^{n+p} - x^n}{lx}$ per solam variabilem x differentiata et per ∂x divisa, si loco x scribatur 1, producet hunc valorem $np + \frac{1}{2}pp$, id quod neutiquam tam facile perspicitur. Si enim illa quantitas differentietur, omisso elemento ∂x pervenitur ad istam expressionem

$$\frac{(n+p)x^{n+p-1} - nx^{n-1}}{lx} - \frac{x^{n+p-1} - x^{n-1}}{(lx)^2},$$

ubi iam poni oportet $x = 1$; tum autem utrumque membrum evadit infinitum, quamobrem has duas fractiones ante omnia ad eundem denominatorem reduci convenit, ut habeatur ista fractio

$$\frac{(n+p)x^{n+p-1}lx - nx^{n-1}lx - x^{n+p-1} + x^{n-1}}{(lx)^2},$$

cuius tam numerator quam denominator evanescunt facto $x=1$. Quamobrem secundum regulam cognitam loco tam numeratoris quam denominatoris eorum differentialia scribantur ac pro numeratore reperietur

$$(n+p)(n+p-1)x^{n+p-2}lx+(n+p)x^{n+p-2}-n(n-1)x^{n-2}lx-nx^{n-2}$$
$$-(n+p-1)x^{n+p-2}+(n-1)x^{n-2};$$

denominator vero erit $\frac{2lx}{x}$, ita ut iam tota fractio sit

$$\frac{(n+p)(n+p-1)x^{n+p-1}lx+(n+p)x^{n+p-1}-n(n-1)x^{n-1}lx-(n+p-1)x^{n+p-1}-x^{n-1}}{2lx},$$

ubi denuo posito $x=1$ tam numerator quam denominator evanescunt; quamobrem eorum loco iterum differentialia substituamus, quo facto prodibit fractio, cuius numerator erit

$$(n+p-1)^2x^{n+p-2}((n+p)lx-1)+2(n+p)(n+p-1)x^{n+p-2}$$
$$-n(n-1)^2x^{n-2}lx-(nn-1)x^{n-2},$$

denominator vero erit $\frac{2}{x}$. Hic iam facto $x=1$ numerator dabit

$$2(n+p)(n+p-1)-(n+p-1)^2-(nn-1)=2np+pp,$$

denominator vero 2, unde valor quaesitus resultat $np+\frac{1}{2}pp$, prorsus uti supra invenimus. Hinc igitur abunde patet egregius usus nostrae reductionis. Quin etiam casus adhuc simplicior, quo $\mu=0$, haud exiguam moram creat.

EXEMPLUM 2 QUO $\mu=0$ ET $\nu=1$

Hic erit $M=1$ ob $Q=x^{n+p}$ manente $R=\frac{x^{n+p}-x^n}{lx}$; tum erit $\frac{\int}{p}\cdot M=p$, unde aequatio more solito expressa fiet

$$\frac{x^{n+p}-x^n}{lx}=p.$$

Posito autem $x=1$ in parte sinistra tam numerator quam denominator evanescunt, unde eorum differentialibus substitutis ista fractio evadet

$$\frac{(n+p)x^{n+p-1}-nx^{n-1}}{1:x},$$

quae fractio posito $x=1$ praebet p.

EXEMPLUM 3 QUO $\mu=0$ ET $\nu=2$

Hic ergo erit $M=1$ ideoque $\frac{\int^2}{p}\cdot M=\frac{1}{2}pp$, cui ergo ipsa quantitas R aequabitur; sicque orietur haec aequatio

$$\frac{x^{n+p}-x^n}{(lx)^2}-\frac{px^n}{lx}=\frac{1}{2}pp,$$

cuius veritas neutiquam in oculos incurrit; quamobrem quantitas R ad unicam fractionem reducatur, quae erit

$$\frac{x^{n+p}-x^n-px^nlx}{(lx)^2},$$

quae fractio, si loco numeratoris et denominatoris eorum differentialia substituantur, abit in sequentem

$$\frac{(n+p)x^{n+p}-nx^n-npx^nlx-px^n}{2lx};$$

haec vero fractio eadem operatione instituta reducitur ad hanc

$$\frac{(n+p)^2x^{n+p}-nnx^n-nnpx^nlx-2npx^n}{2},$$

quae expressio posito $x=1$ manifesto abit in $\frac{1}{2}pp$.

EXEMPLUM 4 QUO $\mu=0$ ET $\nu=\nu$

Hic ergo erit $M=1$ ideoque

$$\frac{\int^\nu}{p}\cdot M=\frac{p^\nu}{1\cdot 2\cdot 3\cdots\nu}.$$

Porro vero vidimus esse

$$R=\frac{x^{n+p}}{(lx)^\nu}-x^n\left(\frac{1}{(lx)^\nu}+\frac{p}{(lx)^{\nu-1}}+\cdots+\frac{p^{\nu-1}}{1\cdot 2\cdot 3\cdots(\nu-1)lx}\right)$$

atque haec expressio R ita est comparata, ut posito $x=1$ eius valor futurus sit $\frac{p^\nu}{1\cdot 2\cdot 3\cdots\nu}$.

EXEMPLUM 5 QUO $\mu=1$ ET $\nu=\nu$

Hic ergo erit $M=n+p$ ideoque

$$\frac{\int^{\nu}}{p}\cdot M=\frac{n(\nu+1)p^{\nu}+p^{\nu+1}}{1\cdot 2\cdot 3\cdots(\nu+1)}.$$

Quodsi iam ponatur

$$R=\frac{x^{n+p}}{(lx)^{\nu}}-x^{n}\left(\frac{1}{(lx)^{\nu}}+\frac{p}{(lx)^{\nu-1}}+\frac{pp}{1\cdot 2(lx)^{\nu-2}}+\cdots+\frac{p^{\nu-1}}{1\cdot 2\cdots(\nu-1)lx}\right),$$

quae expressio ut functio solius x spectetur, tum posito $x=1$ erit more solito

$$\left(\frac{\partial R}{\partial x}\right)=\frac{p^{\nu}(n(\nu+1)+p)}{1\cdot 2\cdot 3\cdots(\nu+1)}.$$

Ubi facile intelligitur differentiale ipsius R formulam producere multo magis complicatam, cuius omnibus terminis ad communem denominatorem reductis, qui erit $(lx)^{\nu+1}$, si per regulam vulgarem istius fractionis valorem casu $x=1$ explorare vellemus, tum tam numerator quam denominator $\nu+1$ vicibus differentiari deberent, antequam eius verus valor definiri posset, quem tamen nunc certe novimus fore $\frac{p^{\nu}(n(\nu+1)+p)}{1\cdot 2\cdot 3\cdots(\nu+1)}$.

EXEMPLUM 6 QUO $\mu=2$ ET $\nu=\nu$

Hic ergo erit

$$M=(n+p)(n+p-1)=n(n-1)+(2n-1)p+pp$$

ideoque

$$\frac{\int^{\nu}}{p}\cdot M=\frac{n(n-1)p^{\nu}}{1\cdot 2\cdot 3\cdots\nu}+\frac{(2n-1)p^{\nu+1}}{2\cdot 3\cdot 4\cdots(\nu+1)}+\frac{p^{\nu+2}}{3\cdot 4\cdot 5\cdots(\nu+2)};$$

tum igitur, si ut ante fuerit

$$R=\frac{x^{n+p}}{(lx)^{\nu}}-x^{n}\left(\frac{1}{(lx)^{\nu}}+\frac{p}{(lx)^{\nu-1}}+\frac{pp}{1\cdot 2(lx)^{\nu-2}}+\cdots+\frac{p^{\nu-1}}{1\cdot 2\cdots(\nu-1)lx}\right),$$

casu $x=1$ erit

$$\left(\frac{\partial\partial R}{\partial x^{2}}\right)=\frac{n(n-1)p^{\nu}}{1\cdot 2\cdot 3\cdots\nu}+\frac{(2n-1)p^{\nu+1}}{2\cdot 3\cdot 4\cdots(\nu+1)}+\frac{p^{\nu+2}}{3\cdot 4\cdot 5\cdots(\nu+2)},$$

quam veritatem more consueto evolvere nemo certe susceperit. Atque ex his iam facile apparet, quomodo has conclusiones pro maioribus valoribus indicis μ formari oporteat.

PROBLEMA 4

Si V fuerit functio quaecunque binarum variabilium x et p et omnes operationes in theoremate quarto indicatae absolvantur, tum vero statuatur $x=1$, exhibere aequalitatem, ad quam hoc theorema perducit.

SOLUTIO

Quoniam in nostro theoremate quarto posuimus $Q=\frac{\int^\mu}{x}\cdot V$, qui valor posito $x=1$ abeat in M, ita ut M futura sit sola functio ipsius p, tum vero $R=\frac{\int^\nu}{p}\cdot V$, vi nostri theorematis semper erit

$$\frac{\int^\mu}{x}\cdot R=\frac{\int^\nu}{p}\cdot M,$$

siquidem omnes integrationes ita absolvantur, ut singula integralia evanescant posito sive $x=0$ sive $p=0$, omnibus autem operationibus peractis statuatur $x=1$. Quodsi iam pro V accipiamus hanc functionem x^{n+p}, primo valores litterae M pro variis indicibus μ sequenti modo se habebunt:

1. Si $\mu=0$, erit $M=1$,

2. si $\mu=1$, erit $M=\frac{1}{n+p+1}$,

3. si $\mu=2$, erit $M=\frac{1}{(n+p+1)(n+p+2)}$,

4. si $\mu=3$, erit $M=\frac{1}{(n+p+1)(n+p+2)(n+p+3)}$

etc.

Hi autem valores ipsius M ope propositionis supra allegatae, quae erat

$$M=\frac{1}{1\cdot2\cdot3\cdots(\mu-1)}\left\{\begin{matrix}\frac{1}{n+p+1}-\frac{\mu-1}{n+p+2}+\frac{(\mu-1)(\mu-2)}{1\cdot2(n+p+3)}\\ -\frac{(\mu-1)(\mu-2)(\mu-3)}{1\cdot2\cdot3(n+p+4)}+\text{etc.}\end{matrix}\right\},$$

sequenti modo pro variis valoribus indicis μ se habebunt:

1. Si $\mu=0$, valor $M=1$,

2. si $\mu=1$, „ $M=\frac{1}{n+p+1}$,

3. si $\mu=2$, „ $M=\frac{1}{n+p+1}-\frac{1}{n+p+2}$,

4. si $\mu=3$, „ $M=\frac{1}{2}\left(\frac{1}{n+p+1}-\frac{2}{n+p+2}+\frac{1}{n+p+3}\right)$,

5. si $\mu=4$, „ $M=\frac{1}{6}\left(\frac{1}{n+p+1}-\frac{3}{n+p+2}+\frac{3}{n+p+3}-\frac{1}{n+p+4}\right)$,

6. si $\mu=5$, „ $M=\frac{1}{24}\left(\frac{1}{n+p+1}-\frac{4}{n+p+2}+\frac{6}{n+p+3}-\frac{4}{n+p+4}+\frac{1}{n+p+5}\right)$

etc.

Deinde pro littera R, si indici ν successive tribuantur valores 0, 1, 2, 3, 4 etc., reperietur,

1. si $\nu=0$, fore $R=x^{n+p}$,

2. si $\nu=1$, „ $R=\frac{x^{n+p}-x^n}{lx}$,

3. si $\nu=2$, „ $R=\frac{x^{n+p}-x^n}{(lx)^2}-\frac{px^n}{lx}$,

4. si $\nu=3$, „ $R=\frac{x^{n+p}-x^n}{(lx)^3}-\frac{px^n}{(lx)^2}-\frac{ppx^n}{2lx}$,

5. si $\nu=4$, „ $R=\frac{x^{n+p}-x^n}{(lx)^4}-\frac{px^n}{(lx)^3}-\frac{ppx^n}{2(lx)^2}-\frac{p^3x^n}{6lx}$

etc.

Hinc igitur sequentia exempla evolvamus.

EXEMPLUM 1 QUO $\mu=0$ ET $\nu=0$

Hoc casu erit

$$M=1 \quad \text{et} \quad R=x^{n+p},$$

unde facto $x=1$ erit utique

$$x^{n+p}=1.$$

EXEMPLUM 2 QUO $\mu=0$ ET $\nu=1$

Hoc ergo casu erit

$$M=1 \quad \text{et} \quad R=\frac{x^{n+p}-x^n}{lx},$$

unde posito $x=1$ fiet

$$\frac{x^{n+p}-x^n}{lx}=p.$$

EXEMPLUM 3 QUO $\mu=0$ ET $\nu=2$

Hoc ergo casu adhuc est

$$M=1 \quad \text{et} \quad R=\frac{x^{n+p}-x^n}{(lx)^2}-\frac{px^n}{lx}.$$

Hinc ergo posito $x=1$ prodibit ista aequalitas

$$\frac{x^{n+p}}{(lx)^2}-x^n\left(\frac{1}{(lx)^2}+\frac{p}{lx}\right)=\frac{pp}{2}.$$

EXEMPLUM 4 QUO $\mu=0$ ET $\nu=3$

Hic ergo manente $M=1$ erit

$$R=\frac{x^{n+p}}{(lx)^3}-x^n\left(\frac{1}{(lx)^3}+\frac{p}{(lx)^2}+\frac{pp}{2lx}\right);$$

quare posito $x=1$ habebitur ista aequatio

$$\frac{x^{n+p}}{(lx)^3}-x^n\left(\frac{1}{(lx)^3}+\frac{p}{(lx)^2}+\frac{pp}{2lx}\right)=\frac{p^3}{6}.$$

Haec autem exempla iam in praecedente problemate occurrunt, quia signa $\int^0$ et ∂^0 aequivalent.

EXEMPLUM 5 QUO $\mu=1$ ET $\nu=1$

Hoc casu erit

$$M=\frac{1}{n+p+1} \quad \text{et} \quad R=\frac{x^{n+p}-x^n}{lx},$$

unde, cum fiat $\int R\partial x = \int M\partial p$, erit

$$\int \frac{x^{n+p}-x^n}{lx}\,\partial x = l\,\frac{n+p+1}{n+1},$$

quod est illud ipsum theorema, quod non ita pridem inveneram et Geometris proposueram.[1])

EXEMPLUM 6 QUO $\mu = 2$ ET $\nu = 1$

Hoc casu erit

$$M = \frac{1}{n+p+1} - \frac{1}{n+p+2}$$

manente

$$R = \frac{x^{n+p}-x^n}{lx}.$$

Hinc igitur posito $x = 1$ oritur ista aequatio

$$\int \partial x \int \frac{x^{n+p}-x^n}{lx}\,\partial x = l\,\frac{n+p+1}{n+1} - l\,\frac{n+p+2}{n+2};$$

haec autem veritas haud difficulter ex praecedente exemplo deduci potest. Cum enim in genere sit

$$\int \partial x \int R\partial x = x\int R\partial x - \int Rx\partial x$$

ideoque casu $x = 1$

$$\int \partial x \int R\partial x = \int R\partial x - \int Rx\partial x,$$

ob $R = \frac{x^{n+p}-x^n}{lx}$ erit ex exemplo praecedente

$$\int R\partial x = l\,\frac{n+p+1}{n+1}$$

atque indidem loco n scribendo $n+1$ erit

$$\int Rx\partial x = l\,\frac{n+p+2}{n+2}$$

sicque ipse valor inventus prodit.

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1) Vide notam p. 336. A. L.

EXEMPLUM 7 QUO $\mu=3$ ET $\nu=1$

Hoc ergo casu erit

$$M=\frac{1}{2}\left(\frac{1}{n+p+1}-\frac{2}{n+p+2}+\frac{1}{n+p+3}\right)$$

hincque

$$\int M\partial p=\frac{1}{2}\,l\frac{n+p+1}{n+1}-\frac{2}{2}\,l\frac{n+p+2}{n+2}+\frac{1}{2}\,l\frac{n+p+3}{n+3};$$

at pro R habetur adhuc valor praecedens $R=\frac{x^{n+p}-x^n}{lx}$. Quare cum per propositionem supra allatam sit

$$\int\partial x\int\partial x\int R\partial x=\int\frac{R\partial x(1-x)^2}{1\cdot 2},$$

habebimus per simplex signum summatorium

$$\int\frac{(1-x)^2(x^{n+p}-x^n)}{lx}\partial x=l\frac{n+p+1}{n+1}-2\,l\frac{n+p+2}{n+2}+l\frac{n+p+3}{n+3}.$$

EXEMPLUM 8 QUO $\mu=4$ ET $\nu=1$

Hoc casu erit

$$M=\frac{1}{6}\left(\frac{1}{n+p+1}-\frac{3}{n+p+2}+\frac{3}{n+p+3}-\frac{1}{n+p+4}\right)$$

hincque

$$\int M\partial p=\frac{1}{6}\,l\frac{n+p+1}{n+1}-\frac{3}{6}\,l\frac{n+p+2}{n+2}+\frac{3}{6}\,l\frac{n+p+3}{n+3}-\frac{1}{6}\,l\frac{n+p+4}{n+4}.$$

Deinde, cum ut ante sit $R=\frac{x^{n+p}-x^n}{lx}$, ob

$$\int\partial x\int\partial x\int\partial x\int R\partial x=\frac{1}{6}\int R\partial x(1-x)^3$$

erit

$$\int\frac{(1-x)^3(x^{n+p}-x^n)}{lx}\partial x=l\frac{n+p+1}{n+1}-3l\frac{n+p+2}{n+2}+3l\frac{n+p+3}{n+3}-l\frac{n+p+4}{n+4}.$$

Superfluum autem foret indici μ maiores valores tribuere, cum facta evolutione formulae $(1-x)^{\mu-1}$ ex exemplo quinto iidem valores essentondituri.

EXEMPLUM 9 QUO $\mu=1$ ET $\nu=2$

Hoc ergo casu erit

$$M=\frac{1}{n+p+1}$$

hincque

$$\int M\partial p=l\frac{n+p+1}{n+1} \quad \text{et} \quad \int\partial p\int M\partial p=(n+p+1)l\frac{n+p+1}{n+1}-p.$$

Facilius autem hic valor reperitur ope reductionis generalis

$$\int\partial p\int M\partial p=p\int M\partial p-\int Mp\partial p;$$

namque ob $M=\frac{1}{n+p+1}$ erit

$$\int M\partial p=l\frac{n+p+1}{n+1},$$

deinde vero ob

$$Mp=\frac{p}{n+p+1}=1-\frac{n+1}{n+p+1}$$

erit

$$\int Mp\,\partial p=p-(n+1)l\frac{n+p+1}{n+1},$$

unde colligitur

$$\int\partial p\int M\partial p=pl\frac{n+p+1}{n+1}+(n+1)l\frac{n+p+1}{n+1}-p$$

ut ante. Tum vero erit

$$R=\frac{x^{n+p}}{(lx)^2}-x^n\left(\frac{1}{(lx)^2}+\frac{p}{lx}\right);$$

hinc, cum sit $\int R\partial x=\int\partial p\int M\partial p$, erit

$$\int\frac{x^{n+p}}{(lx)^2}\partial x-\int x^n\left(\frac{1}{(lx)^2}+\frac{p}{lx}\right)\partial x=(n+p+1)l\frac{n+p+1}{n+1}-p.$$

EXEMPLUM 10 QUO $\mu=2$ ET $\nu=2$

Hoc ergo casu erit

$$M=\frac{1}{n+p+1}-\frac{1}{n+p+2}$$

hincque

$$\int M\partial p=l\frac{n+p+1}{n+1}-l\frac{n+p+2}{n+2}$$

et ob superiorem reductionem hinc fit

$$Mp=\frac{p}{n+p+1}-\frac{p}{n+p+2}=-\frac{n+1}{n+p+1}+\frac{n+2}{n+p+2}$$

ideoque

$$\int Mp\partial p=-(n+1)l\frac{n+p+1}{n+1}+(n+2)l\frac{n+p+2}{n+2},$$

ita ut iam sit

$$\int\partial p\int M\partial p=pl\frac{n+p+1}{n+1}-pl\frac{n+p+2}{n+2}+(n+1)l\frac{n+p+1}{n+1}-(n+2)l\frac{n+p+2}{n+2};$$

quare, cum sit

$$\int\partial x\int R\partial x=\int\partial p\int M\partial p,$$

ob

$$\int\partial x\int R\partial x=\int R\partial x-\int Rx\partial x$$

aequatio hinc oriunda fiet

$$\int\frac{(1-x)x^{n+p}\partial x}{(lx)^2}-\int(1-x)x^n\left(\frac{1}{(lx)^2}+\frac{p}{lx}\right)\partial x$$
$$=(n+p+1)l\frac{n+p+1}{n+1}-(n+p+2)l\frac{n+p+2}{n+2}.$$

EXEMPLUM 11 QUO $\mu=3$ ET $\nu=2$

Hoc ergo casu est

$$M=\frac{1}{2}\left(\frac{1}{n+p+1}-\frac{2}{n+p+2}+\frac{1}{n+p+3}\right),$$

hinc

$$\int M\partial p=\frac{1}{2}l\frac{n+p+1}{n+1}-\frac{2}{2}l\frac{n+p+2}{n+2}+\frac{1}{2}l\frac{n+p+3}{n+3},$$

tum vero

$$Mp = -\frac{\frac{1}{2}(n+1)}{n+p+1} + \frac{\frac{2}{2}(n+2)}{n+p+2} - \frac{\frac{1}{2}(n+3)}{n+p+3}$$

ideoque

$$\int Mp\partial p = -\frac{1}{2}(n+1)\,l\frac{n+p+1}{n+1} + \frac{2}{2}(n+2)\,l\frac{n+p+2}{n+2} - \frac{1}{2}(n+3)\,l\frac{n+p+3}{n+3};$$

consequenter

$$\int\partial p\int M\partial p = \left\{\begin{aligned} &+\frac{1}{2}(n+p+1)\,l\frac{n+p+1}{n+1}\\ &-\frac{2}{2}(n+p+2)\,l\frac{n+p+2}{n+2}\\ &+\frac{1}{2}(n+p+3)\,l\frac{n+p+3}{n+3}\end{aligned}\right\}.$$

Deinde vero manente R ut ante, quoniam sumto $x=1$ in genere est

$$\int\partial x\int\partial x\int R\partial x = \tfrac{1}{2}\int R\partial x(1-x)^2,$$

hinc resultabit sequens aequatio

$$\int\frac{(1-x)^2x^{n+p}\partial x}{(lx)^2} - \int(1-x)^2x^n\left(\frac{1}{(lx)^2}+\frac{p}{lx}\right)\partial x$$

$$= \left\{\begin{aligned} &+\frac{1}{2}(n+p+1)\,l\frac{n+p+1}{n+1}\\ &-\frac{2}{2}(n+p+2)\,l\frac{n+p+2}{n+2}\\ &+\frac{1}{2}(n+p+3)\,l\frac{n+p+3}{n+3}\end{aligned}\right\}.$$

EXEMPLUM 12 QUO $\mu=1$ ET $\nu=3$

Hoc igitur casu erit

$$M = \frac{1}{n+p+1},$$

et quia in genere est

$$\int\partial p\int\partial p\int M\partial p = \tfrac{1}{2}pp\int M\partial p - \tfrac{2}{2}p\int Mp\partial p + \tfrac{1}{2}\int Mpp\partial p,$$

habebimus

$$\int M\partial p = l\frac{n+p+1}{n+1},$$

$$\int Mp\partial p = p-(n+1)\,l\frac{n+p+1}{n+1}$$

et

$$\int Mpp\partial p = \tfrac{1}{2}pp-(n+1)p+(n+1)^2\,l\frac{n+p+1}{n+1};$$

ex his colligitur

$$\int\partial p\int\partial p\int M\partial p = \tfrac{1}{2}(n+p+1)^2\,l\frac{n+p+1}{n+1}-\tfrac{3}{4}pp-\tfrac{1}{2}(n+1)p.$$

Deinde erit hic

$$R=\frac{x^{n+p}}{(lx)^3}-x^n\left(\frac{1}{(lx)^3}+\frac{p}{(lx)^2}+\frac{pp}{2lx}\right).$$

Hinc igitur resultat sequens aequatio

$$\int\frac{x^{n+p}\partial x}{(lx)^3}-\int x^n\left(\frac{1}{(lx)^3}+\frac{p}{(lx)^2}+\frac{pp}{2lx}\right)\partial x$$
$$=\tfrac{1}{2}(n+p+1)^2\,l\frac{n+p+1}{n+1}-\tfrac{3}{4}pp-\tfrac{1}{2}(n+1)p.$$

EXEMPLUM 13 QUO $\mu=2$ ET $\nu=3$

Cum hoc casu sit

$$M=\frac{1}{n+p+1}-\frac{1}{n+p+2},$$

ob

$$\int\partial p\int\partial p\int M\partial p=\tfrac{1}{2}pp\int M\partial p-\tfrac{2}{2}p\int Mp\partial p+\tfrac{1}{2}\int Mpp\partial p$$

quaeratur

$$\int M\partial p=l\frac{n+p+1}{n+1}-l\frac{n+p+2}{n+2}.$$

Porro ob

$$Mp=-\frac{n+1}{n+p+1}+\frac{n+2}{n+p+2}$$

erit

$$\int Mp\partial p=-(n+1)\,l\frac{n+p+1}{n+1}+(n+2)\,l\frac{n+p+2}{n+2}$$

et

$$\int Mpp\partial p=-(n+1)p+(n+1)^2\,l\frac{n+p+1}{n+1}+(n+2)p-(n+2)^2\,l\frac{n+p+2}{n+2},$$

unde fit

$$\int \partial p \int \partial p \int M \partial p = \frac{1}{2}(n+p+1)^2 l \frac{n+p+1}{n+1} - \frac{1}{2}(n+p+2)^2 l \frac{n+p+2}{n+2} + \frac{1}{2}p.$$

Deinde manente R ut supra erit $\int \partial x \int R \partial x = \int R \partial x (1-x)$, unde colligimus

$$\int \frac{(1-x)x^{n+p}\partial x}{(lx)^3} - \int (1-x)x^n \left(\frac{1}{(lx)^3} + \frac{p}{(lx)^2} + \frac{pp}{2lx}\right)\partial x$$
$$= \frac{1}{2}(n+p+1)^2 l \frac{n+p+1}{n+1} - \frac{1}{2}(n+p+2)^2 l \frac{n+p+2}{n+2} + \frac{1}{2}p.$$

SCHOLION

Ad illustranda haec problemata loco V alia functione determinata praeter $V = x^{n+p}$ uti non licuit, propterea quod alia huiusmodi forma non constat, cuius omnium ordinum integralia ex variabilitate ipsius x oriunda re ipsa exhiberi eorumque valores casu $x=1$ dari queant. Hic enim ob nullum plane usum memorabilem reiici convenit tales formas $V = X + P$ et $V = XP$, ubi X significaret functionem ipsius X tantum, P vero ipsius p tantum. Sin autem in unica integratione ex sola variabili x nata acquiescere velimus, praeter formulam hactenus tractatam x^{n+p} etiam duae sequentes in usum vocari possunt

$$V = \frac{x^{n+p-1} + x^{n-p-1}}{1+x^{2n}} \quad \text{et} \quad V = \frac{x^{n+p-1} - x^{n-p-1}}{1-x^{2n}},$$

quandoquidem ostendi utroque casu valorem integralis $\frac{\int}{x}\cdot V$ sive $\int V\partial x$ casu, quo ponitur $x=1$, admodum commode per functionem solius p exprimi posse, postquam scilicet integrale ita fuerit sumtum, ut evanescat posito $x=0$. Iam dudum enim demonstravi*) sub his conditionibus fore

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*) Videatur Dissertatio Ill. EULERI: *De valore formulae integralis*

$$\int \frac{z^{m-1} \pm z^{n-m-1}}{1 \pm z^n}\partial z$$

casu, quo post integrationem ponitur $z=1$. Nov. Comment. T. XIX.[1])

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1) Quae dissertatio est Commentatio 462 (indicis ENESTROEMIANI), *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 358. A. L.

$$\text{I.}\quad \int \frac{x^{n+p-1}+x^{n-p-1}}{1+x^{2n}}\,\partial x = \frac{\pi}{2n\cos.\frac{\pi p}{2n}},$$

$$\text{II.}\quad \int \frac{x^{n+p-1}-x^{n-p-1}}{1-x^{2n}}\,\partial x = -\frac{\pi}{2n}\operatorname{tang.}\frac{\pi p}{2n}.$$

Quamobrem operae pretium erit bina problemata II et IV etiam per has formulas illustrare. Ex utroque scilicet problemate sumto indice $\mu = 1$ primo deduximus $Q = \frac{\int}{x}\cdot V$, tum vero posito $x = 1$ fecimus $Q = M$, unde casu formulae prioris perpetuo erit

$$M = \frac{\pi}{2n\cos.\frac{\pi p}{2n}},$$

casu posterioris formulae

$$M = -\frac{\pi}{2n}\operatorname{tang.}\frac{\pi p}{2n}.$$

Pro altera autem littera R in problemate secundo erat $R = \frac{\partial^\nu}{p}\cdot V$, unde pro formula prima casu $\nu = 1$ erit

$$R = \frac{x^{n+p-1}-x^{n-p-1}}{1+x^{2n}}\,lx$$

et pro posteriore

$$R = \frac{x^{n+p-1}+x^{n-p-1}}{1-x^{2n}}\,lx.$$

Deinde vero sumto $\nu = 2$ erit pro priore formula

$$R = \frac{x^{n+p-1}+x^{n-p-1}}{1+x^{2n}}\,(lx)^2$$

et pro posteriore

$$R = \frac{x^{n+p-1}-x^{n-p-1}}{1-x^{2n}}\,(lx)^2.$$

Simili modo sumto $\nu = 3$ erit pro priore formula

$$R = \frac{x^{n+p-1}-x^{n-p-1}}{1+x^{2n}}\,(lx)^3,$$

pro posteriore vero

$$R = \frac{x^{n+p-1}+x^{n-p-1}}{1-x^{2n}}\,(lx)^3.$$

Atque adeo in genere pro omni indice ν erit pro priore forma

$$R = \frac{x^{n+p-1} \pm x^{n-p-1}}{1+x^{2n}} (lx)^{\nu},$$

pro posteriore vero

$$R = \frac{x^{n+p-1} \mp x^{n-p-1}}{1-x^{2n}} (lx)^{\nu}.$$

Ubi signa superiora valent, si ν numerus par, inferiora vero, si impar.

Pro quarto autem problemate, ubi quantitas R per integrationes definiri debet, cum sit $R = \frac{\int^{\nu}}{p} \cdot V$, reperimus sumto $\nu = 1$ pro priore formula

$$R = \frac{x^{n+p-1} - x^{n-p-1}}{(1+x^{2n})\, lx},$$

pro posteriore vero formula reperitur

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1}}{(1-x^{2n})\, lx}.$$

Sumto autem $\nu = 2$ habebimus pro formula priore

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1}}{(1+x^{2n})(lx)^2},$$

pro posteriore vero

$$R = \frac{x^{n+p-1} - x^{n-p-1}}{(1-x^{2n})(lx)^2} - \frac{2x^{n-1}p}{(1-x^{2n})\, lx}$$

sive

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2x^{n-1}plx}{(1-x^{2n})(lx)^2}.$$

Deinde vero sumto $\nu = 3$ erit pro priore formula

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2px^{n-1}lx}{(1+x^{2n})(lx)^3}$$

et pro posteriore formula

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1} - p^2x^{n-1}(lx)^2}{(1-x^{2n})(lx)^3}.$$

Sumatur porro $\nu = 4$ ac reperiemus pro formula priore

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2x^{n-1} - ppx^{n-1}(lx)^2}{(1+x^{2n})(lx)^4},$$

pro posteriore vero

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2px^{n-1}lx - \frac{1}{3}p^3x^{n-1}(lx)^3}{(1-x^{2n})(lx)^4}.$$

Sumatur porro $\nu = 5$ ac habebimus pro priore formula

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2px^{n-1}lx - \frac{1}{3}p^3x^{n-1}(lx)^3}{(1+x^{2n})(lx)^5}$$

et [pro posteriore]

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1} - ppx^{n-1}(lx)^2 - \frac{1}{12}p^4x^{n-1}(lx)^4}{(1-x^{2n})(lx)^5}.$$

Sit $\nu = 6$ eritque

$$R = \frac{x^{n+p-1} + x^{n-p-1} - 2x^{n-1}\left(1 + \frac{1}{2}p^2(lx)^2 + \frac{1}{24}p^4(lx)^4\right)}{(1+x^{2n})(lx)^6},$$

$$R = \frac{x^{n+p-1} - x^{n-p-1} - 2x^{n-1}\left(plx + \frac{1}{6}p^3(lx)^3 + \frac{1}{120}p^5(lx)^5\right)}{(1-x^{2n})(lx)^6}$$

et hinc lex iam satis elucet, qua sequentes valores progrediuntur.

CONSIDERATIO AEQUATIONIS

$$\int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x} = \frac{\pi}{2n} \sec. \frac{\pi p}{2n}$$

Quodsi hic brevitatis gratia ponamus $M = \frac{\pi}{2n} \sec. \frac{\pi p}{2n}$, primo casu $x = 1$ ex problemate secundo derivantur sequentes aequalitates

$$\text{I. } \int \frac{x^{n+p} - x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}\, lx = \frac{\partial M}{\partial p},$$

$$\text{II. } \int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^2 = \frac{\partial\partial M}{\partial p^2},$$

$$\text{III. } \int \frac{x^{n+p} - x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^3 = \frac{\partial^3 M}{\partial p^3},$$

$$\text{IV. } \int \frac{x^{n+p} + x^{n-p}}{1+x^{2n}} \cdot \frac{\partial x}{x}(lx)^4 = \frac{\partial^4 M}{\partial p^4}$$

etc.

At vero ex problemate quarto prodeunt sequentes aequalitates

$$\text{I.}\quad \int \frac{x^{n+p}-x^{n-p}}{1+x^{2n}}\cdot\frac{\partial x}{x\,lx}=\int M\,\partial p,$$

$$\text{II.}\quad \int \frac{x^{n+p}+x^{n-p}-2x^{n}}{1+x^{2n}}\cdot\frac{\partial x}{x(lx)^2}=\int\partial p\int M\,\partial p,$$

$$\text{III.}\quad \int \frac{x^{n+p}-x^{n-p}-2x^{n}p\,lx}{1+x^{2n}}\cdot\frac{\partial x}{x(lx)^3}=\int\partial p\int\partial p\int M\,\partial p,$$

$$\text{IV.}\quad \int \frac{x^{n+p}+x^{n-p}-2x^{n}\left(1+\frac{1}{2}pp(lx)^2\right)}{1+x^{2n}}\cdot\frac{\partial x}{x(lx)^4}=\int\partial p\int\partial p\int\partial p\int M\,\partial p,$$

$$\text{V.}\quad \int \frac{x^{n+p}-x^{n-p}-2x^{n}\left(p\,lx+\frac{1}{6}p^3(lx)^3\right)}{1+x^{2n}}\cdot\frac{\partial x}{x(lx)^5}=\int\partial p\int\partial p\int\partial p\int\partial p\int M\,\partial p,$$

$$\text{VI.}\quad \int \frac{x^{n+p}+x^{n-p}-2x^{n}\left(1+\frac{1}{2}p^2(lx)^2+\frac{1}{24}p^4(lx)^4\right)}{1+x^{2n}}\cdot\frac{\partial x}{x(lx)^6}=\int\partial p\int\partial p\int\partial p\int\partial p\int\partial p\int M\,\partial p$$

etc.

CONSIDERATIO AEQUATIONIS

$$\int \frac{x^{n+p}-x^{n-p}}{1-x^{2n}}\cdot\frac{\partial x}{x}=-\frac{\pi}{2n}\operatorname{tang.}\frac{\pi p}{2n}$$

Ponamus hic distinctionis gratia $N=-\frac{\pi}{2n}\operatorname{tang.}\frac{\pi p}{2n}$ atque ex problemate secundo nascuntur sequentes aequalitates

$$\text{I.}\quad \int \frac{x^{n+p}+x^{n-p}}{1-x^{2n}}\cdot\frac{\partial x}{x}lx=\frac{\partial N}{\partial p},$$

$$\text{II.}\quad \int \frac{x^{n+p}-x^{n-p}}{1-x^{2n}}\cdot\frac{\partial x}{x}(lx)^2=\frac{\partial\partial N}{\partial p^2},$$

$$\text{III.}\quad \int \frac{x^{n+p}+x^{n-p}}{1-x^{2n}}\cdot\frac{\partial x}{x}(lx)^3=\frac{\partial^3 N}{\partial p^3},$$

$$\text{IV.}\quad \int \frac{x^{n+p}-x^{n-p}}{1-x^{2n}}\cdot\frac{\partial x}{x}(lx)^4=\frac{\partial^4 N}{\partial p^4}$$

etc.

Verum ex theoremate quarto sequentes resultant aequalitates

$$\text{I. } \int \frac{x^{n+p}+x^{n-p}-2x^n}{1-x^{2n}}\cdot\frac{\partial x}{xlx} = \int N\partial p,$$

$$\text{II. } \int \frac{x^{n+p}-x^{n-p}-2x^n plx}{1-x^{2n}}\cdot\frac{\partial x}{x(lx)^2} = \int\partial p\int N\partial p,$$

$$\text{III. } \int \frac{x^{n+p}+x^{n-p}-2x^n\left(1+\frac{1}{2}pp(lx)^2\right)}{1-x^{2n}}\cdot\frac{\partial x}{x(lx)^3} = \int\partial p\int\partial p\int N\partial p,$$

$$\text{IV. } \int \frac{x^{n+p}-x^{n-p}-2x^n\left(plx+\frac{1}{6}p^3(lx)^3\right)}{1-x^{2n}}\cdot\frac{\partial x}{x(lx)^4} = \int\partial p\int\partial p\int\partial p\int N\partial p,$$

$$\text{V. } \int \frac{x^{n+p}+x^{n-p}-2x^n\left(1+\frac{1}{2}pp(lx)^2+\frac{1}{24}p^4(lx)^4\right)}{1-x^{2n}}\cdot\frac{\partial x}{x(lx)^5} = \int\partial p\int\partial p\int\partial p\int\partial p\int N\partial p$$

etc.

In his scilicet formulis quantitates M et N spectantur ut functiones ipsius p atque ex eius variabilitate tam differentiantur quam integrantur.

Ex his igitur abunde intelligitur omnia, quae super hoc argumento a me non ita pridem[1]) sunt prolata, tamquam casus valde particulares in praesenti tractatione contineri.

SCHOLION

Formulae autem istae sequenti modo succinctius exhiberi possunt, ad quas intelligendas notetur in formulis ad sinistram positis valores integralium esse extendendas ab $x=0$ ad $x=1$, in formulis autem ad dextram positis quantitatem p spectari ut variabilem et integralia ita capi, ut evanescant posito $p=0$, tum vero loco $\frac{\pi}{2}$ hic litteram ϱ scribi, ita ut ϱ sit character anguli recti. His igitur praenotatis ex integrali priori

$$\int \frac{x^p+x^{-p}}{x^n+x^{-n}}\cdot\frac{\partial x}{x} = \frac{\varrho}{n}\sec.\frac{p\varrho}{n}$$

per differentiationem sequentia deducuntur

1) Confer Commentationes 463 et 464 voluminis praecedentis. A. L.

$$\text{I.}\quad \int \frac{x^p - x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x}\, lx = \frac{\varrho}{n\partial p}\, \partial \sec. \frac{p\varrho}{n},$$

$$\text{II.}\quad \int \frac{x^p + x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x} (lx)^2 = \frac{\varrho}{n\partial p^2}\, \partial\partial \sec. \frac{p\varrho}{n},$$

$$\text{III.}\quad \int \frac{x^p - x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x} (lx)^3 = \frac{\varrho}{n\partial p^3}\, \partial^3 \sec. \frac{p\varrho}{n},$$

$$\text{IV.}\quad \int \frac{x^p + x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x} (lx)^4 = \frac{\varrho}{n\partial p^4}\, \partial^4 \sec. \frac{p\varrho}{n}$$

etc.,

per integrationem vero sequentes aequalitates oriuntur

$$\text{I.}\quad \int \frac{x^p - x^{-p}}{x^n + x^{-n}} \cdot \frac{\partial x}{x\, lx} = \frac{\varrho}{n} \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{II.}\quad \int \frac{x^p + x^{-p} - 2}{x^n + x^{-n}} \cdot \frac{\partial x}{x(lx)^2} = \frac{\varrho}{n} \int \partial p \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{III.}\quad \int \frac{x^p - x^{-p} - 2plx}{x^n + x^{-n}} \cdot \frac{\partial x}{x(lx)^3} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{IV.}\quad \int \frac{x^p + x^{-p} - 2\left(1 + \frac{1}{2}p^2(lx)^2\right)}{x^n + x^{-n}} \cdot \frac{\partial x}{x(lx)^4} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \sec. \frac{p\varrho}{n},$$

$$\text{V.}\quad \int \frac{x^p - x^{-p} - 2\left(plx + \frac{1}{6}p^3(lx)^3\right)}{x^n + x^{-n}} \cdot \frac{\partial x}{x(lx)^5} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \sec. \frac{p\varrho}{n}$$

etc.

Ex altera autem formula integrali principali

$$\int \frac{x^p - x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} = \frac{\varrho}{n} \operatorname{tang.} \frac{p\varrho}{n}$$

per differentiationem nascuntur sequentes aequationes

$$\text{I.}\quad \int \frac{x^p + x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x}\, lx = \frac{\varrho}{n\partial p}\, \partial \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{II.}\quad \int \frac{x^p - x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} (lx)^2 = \frac{\varrho}{n\partial p^2}\, \partial\partial \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{III.}\quad \int \frac{x^n + x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} (lx)^3 = \frac{\varrho}{n\partial p^3}\, \partial^3 \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{IV.}\quad \int \frac{x^p - x^{-p}}{x^n - x^{-n}} \cdot \frac{\partial x}{x} (lx)^4 = \frac{\varrho}{n\partial p^4}\, \partial^4 \operatorname{tang.} \frac{p\varrho}{n}$$

etc.,

per integrationem vero colliguntur sequentes

$$\text{I. } \int \frac{x^p + x^{-p} - 2}{x^n - x^{-n}} \cdot \frac{\partial x}{x l x} = \frac{\varrho}{n} \int \partial p \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{II. } \int \frac{x^p - x^{-p} - 2plx}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^2} = \frac{\varrho}{n} \int \partial p \int \partial p \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{III. } \int \frac{x^p + x^{-p} - 2\left(1 + \frac{1}{2}p^2(lx)^2\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^3} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{IV. } \int \frac{x^p - x^{-p} - 2\left(plx + \frac{1}{6}p^3(lx)^3\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^4} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \operatorname{tang.} \frac{p\varrho}{n},$$

$$\text{V. } \int \frac{x^p + x^{-p} - 2\left(1 + \frac{1}{2}p^2(lx)^2 + \frac{1}{24}p^4(lx)^4\right)}{x^n - x^{-n}} \cdot \frac{\partial x}{x(lx)^5} = \frac{\varrho}{n} \int \partial p \int \partial p \int \partial p \int \partial p \int \partial p \operatorname{tang.} \frac{p\varrho}{n}$$

etc.

Denique circa omnes has integrationes notari operae erit pretium, si integralia ad sinistram posita a termino $x = 0$ usque ad $x = \infty$ extendantur, tum eorum valores duplo fieri maiores.

INNUMERA THEOREMATA CIRCA FORMULAS INTEGRALES QUORUM DEMONSTRATIO VIRES ANALYSEOS SUPERARE VIDEATUR

Conventui exhibita die 18. Martii 1776

Commentatio 635 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 3—26
Summarium ibidem p. 61—62

SUMMARIUM

Comme ce mémoire n'est pas susceptible d'extrait, vu qu'il ne contient que des formules intégrales, nous nous contenterons d'indiquer la source, où l'immortel Auteur de ce mémoire a puisé le grand nombre de Théorèmes qu'on y trouve exposés et rédigés en quatorze ordres ou classes. Ces Théorèmes sont tous déduits de la considération: que si l'on désigne par la lettre P l'intégrale de la formule $\int V\partial x$, prise depuis le terme $x=0$ jusqu'à un certain terme déterminé $x=k$, comme la variable x n'est plus contenue dans la quantité P, elle peut être regardée comme fonction d'une autre quantité p qui est renfermée en même tems dans la fonction V, et que de là on peut déduire, tant par la différentiation que par l'intégration, une infinité de formules intégrales, toutes comprises entre les mêmes termes d'intégration; par exemple:

Par la différentiation:

$$\int\partial x\left(\frac{\partial V}{\partial p}\right)=\frac{\partial P}{\partial p},$$

$$\int\partial x\left(\frac{\partial\partial V}{\partial p^2}\right)=\frac{\partial\partial P}{\partial p^2},$$

$$\int\partial x\left(\frac{\partial^3 V}{\partial p^3}\right)=\frac{\partial^3 P}{\partial p^3}$$

etc.

Par l'intégration:

$$\int \partial x \int V \partial p = \int P \partial p,$$

$$\int \partial x \int \partial p \int V \partial p = \int \partial p \int P \partial p,$$

$$\int \partial x \int \partial p \int \partial p \int V \partial p = \int \partial p \int \partial p \int P \partial p$$

etc.

en prenant les intégrales

$$\int V \partial p \quad \text{et} \quad \int P \partial p$$

de façon qu'elles évanouissent, en mettant $p = 0$.

C'est d'après ce principe que M. EULER traite, dans ce mémoire, quatorze formules dont il avoit déterminé autrefois les intégrales contenues entre les deux termes d'intégration $x = 0$ et $x = 1$ ou $x = \infty$, dans un mémoire inséré dans le volume précédent des Nouveaux Actes[1]); et il en déduit autant de classes de Théorèmes remarquables dont la démonstration directe paroît être effectivement au dessus des forces de l'Analyse.

Fundamentum horum theorematum in eiusmodi formulis integralibus $\int V \partial x$ est constitutum, quarum valor a termino $x = 0$ usque ad certum terminum definitum $x = k$ per expressionem finitam assignari queat. Quodsi enim istum valorem littera P designemus, ita ut sit

$$\int V \partial x \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = k \end{bmatrix} = P,$$

quoniam ipsa variabilis x in P non amplius inest, ea tamquam functio alius cuiuspiam quantitatis p, quae simul in functione V contineatur, spectari poterit; tum autem sub iisdem integrationis terminis innumerabiles aliae formulae integrales tam per differentiationem quam per integrationem, quemadmodum iam aliquoties fusius exposui[2]), derivari possunt, quae sunt:

1) Voir la note p. 375. A. L.

2) Vide imprimis Commentationem 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421. A. L.

Per differentiationem:

$$\int \partial x \left(\frac{\partial V}{\partial p}\right) = \frac{\partial P}{\partial p},$$

$$\int \partial x \left(\frac{\partial\partial V}{\partial p^2}\right) = \frac{\partial\partial P}{\partial p^2},$$

$$\int \partial x \left(\frac{\partial^3 V}{\partial p^3}\right) = \frac{\partial^3 P}{\partial p^3},$$

$$\int \partial x \left(\frac{\partial^4 V}{\partial p^4}\right) = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem:

$$\int \partial x \int V \partial p = \int P \partial p,$$

$$\int \partial x \int \partial p \int V \partial p = \int \partial p \int P \partial p,$$

$$\int \partial x \int \partial p \int \partial p \int V \partial p = \int \partial p \int \partial p \int P \partial p,$$

$$\int \partial x \int \partial p \int \partial p \int \partial p \int V \partial p = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.,

ubi circa integralia

$$\int V \partial p \quad \text{et} \quad \int P \partial p$$

probe notandum est ea ita capi debere, ut evanescant posito $p = 0$, quod etiam de integrationibus repetitis perpetuo est tenendum.

Cum igitur nuper*) plures huiusmodi generis formulas integrales $\int V \partial x$ in medium attulerim, quarum valores a termino $x = 0$ usque ad terminum vel $x = 1$ vel $x = \infty$ expressione finita assignare licuit, ex qualibet earum formulas integrales tam per differentiationem quam per integrationem inde derivatas conspectui exponam, quas ergo secundum ordinem formularum principalium, ex quibus sunt deductae, hic distinguam.

*) Nova Acta Academiae Sc. Tom. III in Dissertatione: *Methodus facilis inveniendi integrale* etc.[1])

1) Quae dissertatio est Commentatio 620 huius voluminis. A. L.

ORDO PRIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + 2\cos.\theta + x^{-n}} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \frac{\pi \sin.\frac{p}{n}\theta}{n \sin.\theta \sin.\frac{p}{n}\pi}$$

Haec integratio semper locum habet, nisi sit $p - n > 0$; his igitur casibus exceptis ponamus brevitatis gratia

$$P = \frac{\pi \sin.\frac{p}{n}\theta}{n \sin.\theta \sin.\frac{p}{n}\pi};$$

tum vero etiam loco denominatoris $x^n + 2\cos.\theta + x^{-n}$ scribamus $\varDelta$, ita ut iam P spectari possit tamquam functio ipsius p; quibus observatis per differentiationem hinc sequentia theoremata derivantur

$$\text{I. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem inde sequentia theoremata oriuntur

$$\text{I. } \int \frac{x^p - 1}{\varDelta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p - 1 - p l x}{\varDelta} \cdot \frac{\partial x}{x (lx)^2} \begin{bmatrix} \text{ab} & x = 0 \\ \text{ad} & x = \infty \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III.}\quad \int \frac{x^p - 1 - plx - \frac{1}{2}pp(lx)^2}{\varDelta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV.}\quad \int \frac{x^p - 1 - plx - \frac{1}{2}pp(lx)^2 - \frac{1}{6}p^3(lx)^3}{\varDelta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Haecque theoremata aeque subsistunt, sive p sit numerus positivus sive negativus sive etiam integer sive fractus, dumne sit $p - n > 0$ et integralia $\int P\partial p$, $\int \partial p \int P \partial p$, $\int \partial p \int \partial p \int P \partial p$ omniaque hinc deducta ita capiantur, ut evanescant posito $p = 0$.

ORDO SECUNDUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{x^p}{x^{-n}(1 + x^n)^2} \cdot \frac{\partial x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi p}{nn \sin. \frac{p}{n}\pi}$$

Ponamus hic iterum denominatorem $x^{-n}(1 + x^n)^2 = \varDelta$ sitque

$$P = \frac{\pi p}{nn \sin. \frac{p}{n}\pi},$$

ita ut P iterum sit functio ipsius p, ac primo per differentiationem hinc deducentur sequentia theoremata

$$\text{I.}\quad \int \frac{x^p}{\varDelta} \cdot \frac{\partial x lx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II.}\quad \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial\partial P}{\partial p^2},$$

$$\text{III.}\quad \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV.}\quad \int \frac{x^p}{\varDelta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem inde sequentia theoremata oriuntur

$$\text{I. } \int \frac{x^p - 1}{\Delta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p - 1 - plx}{\Delta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - 1 - plx - \frac{1}{2}pp(lx)^2}{\Delta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p - 1 - plx - \frac{1}{2}pp(lx)^2 - \frac{1}{6}p^3(lx)^3}{\Delta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.,

ubi circa integrationes eadem sunt observanda, quae ante fuerant praecepta.

ORDO TERTIUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + \left(f + \frac{1}{f}\right) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi\left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n(f^1 - f^{-1}) \sin . \frac{p}{n}\pi}$$

Ponamus hic iterum pro denominatore

$$\Delta = x^n + \left(f + \frac{1}{f}\right) + x^{-n};$$

tum vero sit

$$P = \frac{\pi\left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n(f^1 - f^{-1}) \sin . \frac{p}{n}\pi} = \frac{\pi\left(f^{1+\frac{p}{n}} - f^{1-\frac{p}{n}}\right)}{n(ff - 1) \sin . \frac{p}{n}\pi}.$$

His positis ut ante per differentiationem sequentia theoremata deducuntur

$$\text{I. } \int \frac{x^p}{\Delta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\partial \partial P}{\partial p^2},$$

$$\text{III.}\quad \int \frac{x^p}{\varDelta}\cdot\frac{\partial x(lx)^3}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV.}\quad \int \frac{x^p}{\varDelta}\cdot\frac{\partial x(lx)^4}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem eliciuntur sequentia

$$\text{I.}\quad \int \frac{x^p-1}{\varDelta}\cdot\frac{\partial x}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\int P\partial p,$$

$$\text{II.}\quad \int \frac{x^p-1-plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^2}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\int\partial p\int P\partial p,$$

$$\text{III.}\quad \int \frac{x^p-1-plx-\frac{1}{2}pp(lx)^2}{\varDelta}\cdot\frac{\partial x}{x(lx)^3}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\int\partial p\int\partial p\int P\partial p,$$

$$\text{IV.}\quad \int \frac{x^p-1-plx-\frac{1}{2}pp(lx)^2-\frac{1}{6}p^3(lx)^3}{\varDelta}\cdot\frac{\partial x}{x(lx)^4}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\int\partial p\int\partial p\int\partial p\int P\partial p$$

etc.

Ubi denuo eadem sunt observanda, quae supra sunt praecepta.

ORDO QUARTUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int\frac{\partial x}{x}\cdot\frac{x^p+x^{-p}}{x^n+2\cos.\theta+x^{-n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\sin.\frac{p}{n}\theta}{n\sin.\theta\sin.\frac{p}{n}\pi}$$

Statuamus hic iterum

$$\varDelta=x^n+2\cos.\theta+x^{-n}$$

sitque

$$P=\frac{\pi\sin.\frac{p}{n}\theta}{n\sin.\theta\sin.\frac{p}{n}\pi},$$

ita ut P tamquam functio ipsius p spectari possit; ubi probe notandum est hunc valorem integralem subsistere non posse, nisi sit $p<n$ ideoque fractio

$\frac{p}{n}$ unitate minor; atque sub iisdem conditionibus per differentiationem sequentia hinc deducuntur theoremata

$$\text{I. } \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem colliguntur sequentia

$$\text{I. } \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p + x^{-p} - 2}{\varDelta} \cdot \frac{\partial x}{x (lx)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - x^{-p} - 2plx}{\varDelta} \cdot \frac{\partial x}{x (lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p + x^{-p} - 2 - \frac{2}{2} pp (lx)^2}{\varDelta} \cdot \frac{\partial x}{x (lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Quodsi eadem integralia extendantur ab $x = 0$ ad $x = \infty$, eorum valores duplo evadent maiores.

ORDO QUINTUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^{-n}(1 + x^n)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi p}{nn \sin. \frac{p}{n} \pi}$$

Statuamus igitur hic pro denominatore

$$\varDelta = x^{-n}(1 + x^n)^2$$

sitque

$$P = \frac{\pi p}{nn \sin . \frac{p}{n} \pi},$$

ita ut P spectari possit tamquam functio ipsius p, ubi perpetuo fractio $\frac{p}{n}$ unitate minor supponitur; quibus positis per differentiationem sequentia nascuntur theoremata

$$\text{I.}\quad \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II.}\quad \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial \partial P}{\partial p^2},$$

$$\text{III.}\quad \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV.}\quad \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem vero sequentia deducuntur

$$\text{I.}\quad \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P \partial p,$$

$$\text{II.}\quad \int \frac{x^p + x^{-p} - 2}{\varDelta} \cdot \frac{\partial x}{x (lx)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III.}\quad \int \frac{x^p - x^{-p} - 2plx}{\varDelta} \cdot \frac{\partial x}{x (lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV.}\quad \int \frac{x^p + x^{-p} - 2 - pp(lx)^2}{\varDelta} \cdot \frac{\partial x}{x (lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

At si haec integralia ab $x = 0$ ad $x = \infty$ capiantur, eorum valores evadent duplo maiores.

ORDO SEXTUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + \left(f + \frac{1}{f}\right) + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi\left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n(f^1 - f^{-1}) \sin. \frac{p}{n}\pi}$$

Statuamus

$$\Delta = x^n + \left(f + \frac{1}{f}\right) + x^{-n} = \frac{1}{x^n}(x^n + f)\left(x^n + \frac{1}{f}\right)$$

et sit

$$P = \frac{\pi\left(f^{\frac{p}{n}} - f^{-\frac{p}{n}}\right)}{n(f^1 - f^{-1}) \sin. \frac{p}{n}\pi},$$

ubi iterum fractio $\frac{p}{n}$ unitate minor supponitur. His observatis per differentiationem colligimus

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x\, lx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p + x^{-p}}{\Delta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem sequentia theoremata nascuntur

$$\text{I. } \int \frac{x^p - x^{-p}}{\Delta} \cdot \frac{\partial x}{x\, lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int P \partial p,$$

$$\text{II. } \int \frac{x^p + x^{-p} - 2}{\Delta} \cdot \frac{\partial x}{x (lx)^2} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p - x^{-p} - 2p\, lx}{\Delta} \cdot \frac{\partial x}{x (lx)^3} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p + x^{-p} - 2 - pp(lx)^2}{\Delta} \cdot \frac{\partial x}{x (lx)^4} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.

Quodsi haec integralia ab $x=0$ ad $x=\infty$ extendantur, eorum valores erunt duplo maiores. Ceterum hic perspicuum est quantitatem f esse debere positivam, quia alias[1]) potestates $f^{\pm\frac{p}{n}}$ fieri possent imaginariae.

ORDO SEPTIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int\frac{\partial x}{x}\cdot\frac{\cos.plx}{x^n+2\cos.\theta+x^{-n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2n\sin.\theta}\cdot\frac{e^{\frac{p}{n}\theta}-e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}}$$

Statuamus hic iterum pro denominatore

$$\varDelta=x^n+2\cos.\theta+x^{-n}$$

sitque

$$P=\frac{\pi}{2n\sin.\theta}\cdot\frac{e^{\frac{p}{n}\theta}-e^{-\frac{p}{n}\theta}}{e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}},$$

quae ergo quantitas iterum ut functio ipsius p spectari potest; ubi autem non amplius necesse est, ut fractio $\frac{p}{n}$ sit unitate minor. Hinc igitur per differentiationem sequentia derivantur theoremata

$$\text{I. }\int\frac{\sin.plx}{\varDelta}\cdot\frac{\partial x\,lx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\partial P}{\partial p},$$

$$\text{II. }\int\frac{\cos.plx}{\varDelta}\cdot\frac{\partial x(lx)^2}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\partial\partial P}{\partial p^2},$$

$$\text{III. }\int\frac{\sin.plx}{\varDelta}\cdot\frac{\partial x(lx)^3}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=+\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. }\int\frac{\cos.plx}{\varDelta}\cdot\frac{\partial x(lx)^4}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=+\frac{\partial^4 P}{\partial p^4}$$

etc.,

1) Si quantitas f est negativa, integralia non valent. A. L.

per integrationem vero

$$\text{I.}\quad \int \frac{\sin. plx}{\varDelta}\cdot\frac{\partial x}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \int P\partial p,$$

$$\text{II.}\quad \int \frac{1-\cos. plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^2}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \int \partial p\int P\partial p,$$

$$\text{III.}\quad \int \frac{plx-\sin. plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^3}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \int \partial p\int \partial p\int P\partial p,$$

$$\text{IV.}\quad \int \frac{\frac{1}{2}pp(lx)^2-1+\cos. plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^4}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \int \partial p\int \partial p\int \partial p\int P\partial p,$$

$$\text{V.}\quad \int \frac{\frac{1}{6}p^3(lx)^3-plx+\sin. plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^5}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \int \partial p\int \partial p\int \partial p\int \partial p\int P\partial p$$

etc.

Haec igitur integralia si ab $x=0$ ad $x=\infty$ extendantur, iterum duplo fiunt maiora.

ORDO OCTAVUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x}\cdot\frac{\cos. plx}{x^{-n}(x^n+1)^2}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{nn}\cdot\frac{p}{e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}}$$

Statuamus hic pro denominatore

$$\varDelta = x^{-n}(x^n+1)^2$$

sitque

$$P = \frac{\pi}{nn}\cdot\frac{p}{e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}}$$

atque per differentiationem hinc deducentur sequentia theoremata

$$\text{I.}\quad \int \frac{\sin. plx}{\varDelta}\cdot\frac{\partial xlx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{\partial P}{\partial p},$$

$$\text{II.}\quad \int \frac{\cos. plx}{\varDelta}\cdot\frac{\partial x(lx)^2}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\frac{\partial\partial P}{\partial p^2},$$

$$\text{III.}\ \int\frac{\sin.plx}{\varDelta}\cdot\frac{\partial x(lx)^3}{x}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=+\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV.}\ \int\frac{\cos.plx}{\varDelta}\cdot\frac{\partial x(lx)^4}{x}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=+\frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem vero elicitur

$$\text{I.}\ \int\frac{\sin.plx}{\varDelta}\cdot\frac{\partial x}{xlx}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=\int P\partial p,$$

$$\text{II.}\ \int\frac{1-\cos.plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^2}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=\int\partial p\int P\partial p,$$

$$\text{III.}\ \int\frac{plx-\sin.plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^3}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=\int\partial p\int\partial p\int P\partial p,$$

$$\text{IV.}\ \int\frac{\frac{1}{2}pp(lx)^2-1+\cos.plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^4}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=\int\partial p\int\partial p\int\partial p\int P\partial p,$$

$$\text{V.}\ \int\frac{\frac{1}{6}p^3(lx)^3-plx+\sin.plx}{\varDelta}\cdot\frac{\partial x}{x(lx)^5}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=\int\partial p\int\partial p\int\partial p\int\partial p\int P\partial p$$

etc.

ORDO NONUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int\frac{\partial x}{x}\cdot\frac{\cos.plx}{x^n+\left(f+\frac{1}{f}\right)+x^{-n}}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=\frac{2\pi\sin.\frac{p}{n}lf}{n\left(f-\frac{1}{f}\right)\left(e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}\right)}$$

Statuatur

$$\varDelta=x^n+\left(f+\frac{1}{f}\right)+x^{-n}$$

sitque

$$P=\frac{2\pi\sin.\frac{p}{n}lf}{n\left(f-\frac{1}{f}\right)\left(e^{\frac{p}{n}\pi}-e^{-\frac{p}{n}\pi}\right)}$$

atque hinc per differentiationem sequentia prodeunt theoremata

$$\text{I.}\quad \int \frac{\sin. plx}{\Delta}\cdot\frac{\partial x lx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\partial P}{\partial p},$$

$$\text{II.}\quad \int \frac{\cos. plx}{\Delta}\cdot\frac{\partial x (lx)^2}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=-\frac{\partial\partial P}{\partial p^2},$$

$$\text{III.}\quad \int \frac{\sin. plx}{\Delta}\cdot\frac{\partial x (lx)^3}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=+\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV.}\quad \int \frac{\cos. plx}{\Delta}\cdot\frac{\partial x (lx)^4}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=+\frac{\partial^4 P}{\partial p^4}$$

etc.,

per integrationem vero

$$\text{I.}\quad \int \frac{\sin. plx}{\Delta}\cdot\frac{\partial x}{xlx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\int P\partial p,$$

$$\text{II.}\quad \int \frac{1-\cos. plx}{\Delta}\cdot\frac{\partial x}{x(lx)^2}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\int\partial p\int P\partial p,$$

$$\text{III.}\quad \int \frac{plx-\sin. plx}{\Delta}\cdot\frac{\partial x}{x(lx)^3}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\int\partial p\int\partial p\int P\partial p,$$

$$\text{IV.}\quad \int \frac{\frac{1}{2}pp(lx)^2-1+\cos. plx}{\Delta}\cdot\frac{\partial x}{x(lx)^4}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\int\partial p\int\partial p\int\partial p\int P\partial p$$

etc.

Hic manifestum est quantitatem f negativam accipi non posse, quia alias[1]) iam ipsa functio P fieret imaginaria.

Adiungamus his theoremata simpliciora, quae ex hactenus allatis nascuntur, dum angulus θ sumitur rectus, ut sit $\cos.\theta=0$ et $\sin.\theta=1$. Hinc ergo sequentes ordines adiiciamus.

1) Vide notam p. 383. A. L.

ORDO DECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

Haec forma scilicet nata est ex prima sumendo $\theta = \frac{\pi}{2}$, unde posito

$$\varDelta = x^n + x^{-n}$$

et

$$P = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

nascuntur eaedem formulae, quae in ordine primo sunt allatae. Hic autem imprimis notari meretur, quod integrale $\int P\partial p$ per logarithmos exhiberi potest; erit enim

$$\int P\partial p = \int \frac{\pi \partial p}{2n \cos. \frac{\pi p}{2n}} = l \text{ tang.} \left(45^0 + \frac{\pi p}{4n}\right),$$

quod integrale ita est sumtum, ut evanescat facto $p = 0$.

ORDO UNDECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p + x^{-p}}{x^n + x^{-n}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

Hic scilicet ordo natus est ex quarto ponendo $\theta = \frac{\pi}{2}$; quamobrem statuamus

$$\varDelta = x^n + x^{-n}$$

et

$$P = \frac{\pi}{2n \cos. \frac{\pi p}{2n}}$$

eademque theoremata inde nascuntur, quae supra pro ordine quarto sunt allata, ubi ergo iterum commode usu venit, ut sit

$$\int P\partial p = l \text{ tang.} \left(45^0 + \frac{\pi p}{4n}\right).$$

ORDO DUODECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x}\cdot\frac{\cos. plx}{x^n+x^{-n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2n}\cdot\frac{1}{e^{\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}}$$

Quodsi ergo statuamus

$$\varDelta = x^n + x^{-n}$$

et

$$P=\frac{\pi}{2n\left(e^{\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}\right)},$$

eadem plane theoremata hinc oriuntur, quae supra pro casu septimo sunt allata. Hic autem iterum notasse iuvabit integrale $\int P\partial p$ revera exhiberi posse. Cum enim sit

$$\int P\partial p=\int\frac{\pi\partial p}{2n\left(e^{\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}\right)},$$

ponatur $\frac{\pi p}{2n}=z$ eritque

$$\int P\partial p=\int\frac{\partial z}{e^z+e^{-z}}=\int\frac{e^z\partial z}{e^{2z}+1}.$$

Sit porro $e^z=v$; erit $\partial v=e^z\partial z$ hincque fiet

$$\int P\partial p=\int\frac{\partial v}{1+vv}=\text{A tang.}\, v;$$

quare retro substituendo habebimus

$$\int P\partial p=\text{A tang.}\, e^{\frac{\pi p}{2n}}.$$

Denique adhuc referamus formulas illas integrales, in quarum denominatore erat $1-x^{2n}$, quas quidem iam olim[1]) breviter tetigi, nunc autem uberius evolvam.

1) Vide L. Euleri Commentationem 462 (indicis Enestroemiani): *De valore formulae integralis* $\int\frac{z^{m-1}\pm z^{n-m-1}}{1\pm z^n}dz$ *casu, quo post integrationem ponitur* $z=1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 3; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 358. A. L.

ORDO TERTIUS DECIMUS
THEOREMATUM EX HAC FORMA PRINCIPALI DEDUCTORUM

$$\int \frac{\partial x}{x} \cdot \frac{x^p - x^{-p}}{x^n - x^{-n}} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \frac{\pi}{2n} \text{ tang. } \frac{\pi p}{2n}$$

Hic igitur iterum statuamus

$$\varDelta = x^n - x^{-n}$$

et

$$P = \frac{\pi}{2n} \text{ tang. } \frac{\pi p}{2n}$$

atque per differentiationem nanciscemur sequentia theoremata

$$\text{I. } \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x l x}{x} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^2}{x} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{x^p + x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^3}{x} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{x^p - x^{-p}}{\varDelta} \cdot \frac{\partial x (lx)^4}{x} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \frac{\partial^4 P}{\partial p^4}$$

etc.

Integratio autem sequentia suppeditat

$$\text{I. } \int \frac{x^p + x^{-p} - 2}{\varDelta} \cdot \frac{\partial x}{x l x} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \int P \partial p,$$

$$\text{II } \int \frac{x^p - x^{-p} - 2plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \int \partial p \int P \partial p,$$

$$\text{III. } \int \frac{x^p + x^{-p} - 2 - \frac{2}{2}pp(lx)^2}{\varDelta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \int \partial p \int \partial p \int P \partial p,$$

$$\text{IV. } \int \frac{x^p - x^{-p} - 2plx - \frac{2}{6}p^3(lx)^3}{\varDelta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P \partial p,$$

$$\text{V. } \int \frac{x^p + x^{-p} - 2 - \frac{2}{2}pp(lx)^2 - \frac{2}{24}p^4(lx)^4}{\varDelta} \cdot \frac{\partial x}{x(lx)^5} \begin{bmatrix} \text{ab} & x=0 \\ \text{ad} & x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int \partial p \int P \partial p$$

etc.,

ubi iterum notetur formulam integralem $\int P\partial p$ actu exhiberi posse; erit enim

$$\int P\partial p = \int \frac{\pi\partial p}{2n}\,\text{tang.}\,\frac{\pi p}{2n} = -\,l\cos.\frac{\pi p}{2n} = +\,l\,\text{sec.}\,\frac{\pi p}{2n}.$$

Hic probe notandum est fractionem $\frac{p}{n}$ semper esse debere unitate minorem.

ORDO QUARTUS DECIMUS
THEOREMATUM EX HAC FORMA GENERALI DEDUCTORUM

$$\int \frac{\partial x}{x}\cdot\frac{\sin. plx}{x^{-n}-x^{+n}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{4n}\cdot\frac{e^{-\frac{p\pi}{2n}}-e^{+\frac{p\pi}{2n}}}{e^{+\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}}$$

Statuatur igitur ut hactenus

$$\varDelta = x^{-n} - x^{n}$$

et

$$P = \frac{\pi}{4n}\cdot\frac{e^{-\frac{p\pi}{2n}}-e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}}+e^{-\frac{p\pi}{2n}}}$$

atque differentiatio nobis praebebit sequentia theoremata

$$\text{I. } \int \frac{\cos. plx}{\varDelta}\cdot\frac{\partial x lx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = +\,\frac{\partial P}{\partial p},$$

$$\text{II. } \int \frac{\sin. plx}{\varDelta}\cdot\frac{\partial x (lx)^2}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\,\frac{\partial\partial P}{\partial p^2},$$

$$\text{III. } \int \frac{\cos. plx}{\varDelta}\cdot\frac{\partial x (lx)^3}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = -\,\frac{\partial^3 P}{\partial p^3},$$

$$\text{IV. } \int \frac{\sin. plx}{\varDelta}\cdot\frac{\partial x (lx)^4}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = +\,\frac{\partial^4 P}{\partial p^4}$$

etc.

Per integrationem autem impetramus sequentia

$$\text{I. } \int \frac{1-\cos. plx}{\varDelta} \cdot \frac{\partial x}{xlx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int P\partial p,$$

$$\text{II. } \int \frac{plx-\sin. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^2} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int P\partial p,$$

$$\text{III. } \int \frac{\frac{1}{2}pp(lx)^2-1+\cos. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^3} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int P\partial p,$$

$$\text{IV. } \int \frac{\frac{1}{6}p^3(lx)^3-plx+\sin. plx}{\varDelta} \cdot \frac{\partial x}{x(lx)^4} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \int \partial p \int \partial p \int \partial p \int P\partial p$$

etc.,

ubi iterum commode evenit, ut $\int P\partial p$ exhiberi possit, siquidem habemus

$$\int P\partial p = \int \frac{\pi\partial p}{4n} \cdot \frac{e^{-\frac{p\pi}{2n}} - e^{+\frac{p\pi}{2n}}}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Ponatur enim $\frac{p\pi}{2n} = \varphi$ eritque

$$\int P\partial p = \int \frac{1}{2}\partial\varphi \cdot \frac{e^{-\varphi}-e^{+\varphi}}{e^{\varphi}+e^{-\varphi}},$$

ubi denominatoris differentiale est $e^{\varphi}\partial\varphi - e^{-\varphi}\partial\varphi$, unde concluditur

$$\int P\partial p = -lV(e^{\varphi}+e^{-\varphi}) + C,$$

quae constans C ita assumi debet, ut integrale evanescat posito $\varphi = 0$, unde fit

$$\int P\partial p = \frac{1}{2}l\frac{2}{e^{\frac{p\pi}{2n}} + e^{-\frac{p\pi}{2n}}}.$$

Hic autem perinde est, utrum fractio $\frac{p}{n}$ maior sit minorve unitate.

COMPARATIO VALORUM FORMULAE INTEGRALIS

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

A TERMINO $x=0$ USQUE AD $x=1$ EXTENSAE

Conventui exhibita die 10. Octobris 1776

Commentatio 640 indicis Enestroemiani
Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 86—117
Summarium ibidem p. 69—72

SUMMARIUM

Si dans la formule exposée dans le titre de ce mémoire, où n, p, q sont des nombres entiers positifs, on donne, pour chaque exposant n, aux p et q toutes les valeurs possibles, il en naît des formules intégrales dont les valeurs ont entre elles des rapports remarquables et tels que, si quelques-unes de ces formules sont connues, on en peut déduire les valeurs de toutes les autres. Feu M. Euler avoit déjà démontré dans son *Calcul Intégral*, Tom. I. Chap. VIII.[1]) plusieurs de ces rapports, mais d'une manière bien éloignée d'épuiser le sujet; il se propose donc ici d'employer une méthode plus féconde, moyennant laquelle on puisse assigner tous les rapports de ce genre et enrichir l'Analyse d'une infinité de nouveaux Théorèmes.

Quelque innombrable que soit la multitude des cas qui paroissent devoir naître, lorsque, pour chaque exposant n, on donne aux p et q toutes les valeurs possibles, on pourra pourtant, quelque grandes que soyent les valeurs de p et q, réduire tous ces cas à d'autres, où les p et q sont diminués de la quantité n, et continuer cette réduction jusqu'à ce que tant p que q soit plus petit que n. Ainsi cette multitude de cas se réduira pour chaque exposant n à un nombre fort modique et déterminé.

1) Voir la note p. 394. A. L.

Le principe qui sert de fondement à toutes ces comparaisons, et que M. EULER n'a pas manqué de démontrer solidement, est que

$$\left[\int \frac{x^{a-1}\partial x}{\sqrt[n]{(1-x^n)^{n-b}}} \text{ ou } \int \frac{x^{b-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a}}}\right] \times \left[\int \frac{x^{a+b-1}\partial x}{\sqrt[n]{(1-x^n)^{n-c}}} \text{ ou } \int \frac{x^{c-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a-b}}}\right]$$
$$= \left[\int \frac{x^{a-1}\partial x}{\sqrt[n]{(1-x^n)^{n-c}}} \text{ ou } \int \frac{x^{c-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a}}}\right] \times \left[\int \frac{x^{a+c-1}\partial x}{\sqrt[n]{(1-x^n)^{n-b}}} \text{ ou } \int \frac{x^{b-1}\partial x}{\sqrt[n]{(1-x^n)^{n-a-c}}}\right],$$

rapport que l'Auteur représente ainsi:

$$(a, b)\,(a+b, c) = (a, c)\,(a+c, b),$$

et ce symbolisme lui facilite infiniment les comparaisons à faire.

M. EULER parcourt donc successivement sept cas différens, en commençant par $a+b=3$ et finissant par $a+b=9$, ce qui lui donne sept classes de valeurs, qui comprennent en tout 39 formules. Ensuite il examine la formule générale proposée selon les différentes valeurs de l'exposant n, depuis $n=3$ jusqu'à $n=7$, ce qui forme cinq ordres différens de rapports, qui fournissent en tout 50 formules intégrales, parmi lesquelles il y a de fort remarquables. On y trouve par exemple les égalités suivantes:[1])

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}},$$
$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} : \int \frac{\partial x}{\sqrt{(1-x^4)}} = \sqrt{2},$$
$$\int \frac{x\partial x}{\sqrt[6]{(1-x^6)^5}} : \int \frac{xx\partial x}{\sqrt[6]{(1-x^6)^4}} = 2.$$

Et quantité d'autres non moins remarquables, toutes pour les termes d'intégration $x=0$ et $x=1$.

Dans chaque ordre il y a donc un certain nombre de formules intégrales qui peuvent être exprimées ou par des quantités circulaires, ou par des quantités transcendantes qui dans un des ordres précédens ont été circulaires, ou bien leurs valeurs seront composées de

1) Editio princeps: *On y trouve par exemple les valeurs des produits suivans:*

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \times \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}};$$
$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} \times \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = \sqrt{2};$$
$$\int \frac{x\partial x}{\sqrt[6]{(1-x^6)^3}} \times \int \frac{xx\partial x}{\sqrt[6]{(1-x^6)^4}} = 2.$$

Corrigé par A. L.

quantités circulaires et transcendantes. Et en regardant comme connues les formules transcendantes qui dans les ordres précédens ont eu des valeurs circulaires, on sera en état de déterminer à leur aide toutes les autres formules contenues dans l'ordre qu'on traite.

Afin de ne rien laisser à désirer dans ces recherches, l'Auteur donne à la fin de son Mémoire une méthode de déterminer, à peu près, les valeurs transcendantes des formules intégrales qu'on est obligé de regarder comme connues dans chaque ordre. Il faut pour cet effet, comme chacun sait, intégrer par série, et pour les mêmes termes d'intégration, la formule intégrale générale annoncée dans le titre du présent mémoire. Mais comme la méthode ordinaire ne fournit pas de série assez convergente, M. EULER s'occupe à remédier à ce défaut en exprimant par deux séries l'intégrale de la formule proposée, l'une depuis $x=0$ jusqu'à $x^n=\frac{1}{2}$, l'autre depuis $x^n=\frac{1}{2}$ jusqu'à $x=1$, dont chacune est très convergente et dont la somme donne la valeur requise pour les termes prescrits d'intégration.

1. In hac formula litterae n, p et q perpetuo designant numeros integros positivos et pro quolibet numero n binis litteris p et q omnes valores tribui concipiuntur, ita ut hinc pro quovis numero n innumerae nascantur huiusmodi formulae integrales, quarum valores plurimas egregias relationes inter se servant; unde si eorum aliquot fuerint cogniti, reliquae omnes ex iis definiri queant. Iam dudum[1]) equidem plures huiusmodi relationes demonstravi; cum autem hoc argumentum tum temporis neutiquam exhausissem, nunc accuratius in istas relationes inquirere constitui et eiusmodi methodum adhibebo, quae omnes plane huius generis relationes sit exhibitura; his enim inventis innumerabilia theoremata condi poterunt, quibus universa Analysis non mediocriter locupletari erit censenda.

2. Quoniam igitur hoc modo pro quolibet numero n ambae litterae p et q infinitos valores recipere possunt, ante omnia hic observari convenit omnes hos innumerabiles casus semper ad numerum finitum revocari posse. Quantumvis enim magni numeri pro litteris p et q accipiantur, eos casus semper

1) Vide L. EULERI Commentationem 321 (indicis ENESTROEMIANI): *Observationes circa integralia formularum* $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ *posito post integrationem* $x=1$, Mélanges de phil. et de mathém. de la société de Turin 3_2 (1762/5), 1766, p. 156; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 268. Vide porro L. EULERI *Institutionum calculi integralis* vol. 1, Petropoli 1768, sectio 1, cap. VIII; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 208. A. L.

ad alios reducere licet, in quibus numeri p et q quantitate n futuri sint diminuti. Hoc igitur modo omnes huiusmodi casus tandem eo redigi poterunt, ut ambo numeri p et q infra exponentem n deprimantur; unde pro quolibet numero n eos tantum casus considerasse sufficiet, quibus litterae p et q minores valores recipiant quam n vel saltem hunc limitem non superent. Hoc igitur modo pro quovis numero n multitudo casuum, qui in computum veniunt et quos inter se comparari oportet, prorsus erit determinata.

3. Quemadmodum autem ista reductio litterarum p et q ad numeros continuo minores institui debeat, quamquam id satis in vulgus est notum, tamen ad formulam praesentem accommodasse iuvabit. Statuatur scilicet haec formula algebraica

$$x^p(1-x^n)^{\frac{q}{n}} = V$$

eritque

$$lV = plx + \frac{q}{n}l(1-x^n),$$

hinc differentiando

$$\frac{\partial V}{V} = \frac{p\partial x}{x} - \frac{qx^{n-1}\partial x}{1-x^n} = \frac{p\partial x - (p+q)x^n\partial x}{x(1-x^n)};$$

ubi si per V multiplicemus ac per partes integremus, orietur ista aequatio

$$V = p\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}} - (p+q)\int x^{p+n-1}\partial x(1-x^n)^{\frac{q-n}{n}}.$$

Quoniam igitur quantitas V pro utroque integrationis termino evanescit, hinc adipiscimur istam reductionem

$$\int x^{p+n-1}\partial x(1-x^n)^{\frac{q-n}{n}} = \frac{p}{p+q}\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}},$$

cuius ergo reductionis ope exponens ipsius x continuo quantitate n diminui poterit, donec tandem infra n deprimatur.

4. Deinde formula pro

$$\frac{\partial V}{V} = \frac{p\partial x - (p+q)x^n\partial x}{x(1-x^n)}$$

inventa hoc modo referri poterit

$$\frac{\partial V}{V}=\frac{(p+q)\partial x(1-x^n)-q\partial x}{x(1-x^n)},$$

quae forma per V multiplicata ac denuo per partes integrata dabit

$$V=(p+q)\int x^{p-1}\partial x(1-x^n)^{\frac{q}{n}}-q\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}},$$

unde, quia posito [$x=0$ et] $x=1$ fit $V=0$, oritur haec reductio

$$\int x^{p-1}\partial x(1-x^n)^{\frac{q}{n}}=\frac{q}{p+q}\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}},$$

cuius reductionis ope exponens binomii $1-x^n$ unitate minuitur sive, quod eodem redit, numerus q numero n imminuitur. Tali igitur reductione, quoties opus fuerit, repetita exponens q tandem infra n deprimi poterit.

5. Quoniam igitur pro quovis numero n ambos exponentes p et q tamquam minores quam n spectare licet, formulam propositam hoc modo expressam repraesentemus

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Hic scilicet pro quovis numero n sufficiet litteris p et q omnes valores ipso n minores tribuisse, quo pacto multitudo omnium casuum ad quemlibet exponentem n pertinentium ad numerum satis modicum reducetur, qui tamen eo maior evadit, quo maior fuerit exponens n.

6. Multo magis autem numerus casuum diversorum diminuetur, si perpendamus ambas litteras p et q inter se permutari posse, ita ut huius formulae

$$\frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

valor ab illo prorsus non discrepet. Ad quod ostendendum ponamus

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=S,$$

si scilicet ista formula integralis ab $x=0$ usque ad $x=1$ extendatur. Iam faciamus

$$1-x^n=y^n,$$

ut formula sit

$$S=\int\frac{x^{p-1}\partial x}{y^{n-q}};$$

tum vero, quia $x^n=1-y^n$, erit $x=(1-y^n)^{\frac{1}{n}}$ hincque

$$x^p=(1-y^n)^{\frac{p}{n}},$$

unde differentiando fit

$$px^{p-1}\partial x=-py^{n-1}\partial y(1-y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$S=-\int y^{q-1}\partial y(1-y^n)^{\frac{p-n}{n}},$$

quam formulam ab $x=0$ usque ad $x=1$, hoc est ab $y=1$ usque ad $y=0$, extendi oportet; permutatis igitur his terminis erit

$$S=\int\frac{y^{q-1}\partial y}{\sqrt[n]{(1-y^n)^{n-p}}}\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=1\end{bmatrix}.$$

Sicque demonstratum est ambas litteras p et q semper inter se esse permutabiles.

7. His praemissis, quo calculos sequentes magis in compendium redigere liceat, loco formulae huius integralis

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

scribamus hunc characterem

$$(p,\ q),$$

ubi perinde est, sive p ante q sive q ante p collocetur; semper autem hic certus exponens n subintelligi debet. Hic autem duo casus prae reliquis maxime memorabiles occurrunt. Prior casus est, quo numerorum p et q

alteruter ipsi exponenti n est aequalis; si enim fuerit $q=n$, erit ex priore formula

$$(p,n)=\int x^{p-1}\partial x=\frac{1}{p}$$

sicque perpetuo habebimus

$$(p,n)=\frac{1}{p}$$

hincque etiam

$$(n,q)=\frac{1}{q}.$$

Alter casus notatu dignissimus locum habet, quando $p+q=n$, quo casu semper est

$$(p,q)=\frac{\pi}{n\sin.\frac{p\pi}{n}}=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

Ad hoc ostendendum sit $q=n-p$ hincque formula proposita

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^p}};$$

tum ponatur

$$\frac{x}{\sqrt[n]{(1-x^n)}}=z,$$

et quia $\frac{x^p}{\sqrt[n]{(1-x^n)^p}}=z^p$, erit

$$S=\int\frac{z^p\partial x}{x}.$$

Ex facta autem positione sequitur

$$x^n=\frac{z^n}{1+z^n}$$

hincque

$$nlx=nlz-l(1+z^n),$$

ergo differentiando

$$\frac{\partial x}{x}=\frac{\partial z}{z}-\frac{z^{n-1}\partial z}{1+z^n}=\frac{\partial z}{z(1+z^n)},$$

ita ut iam sit

$$S=\int\frac{z^{p-1}\partial z}{1+z^n}.$$

Quia autem sumpto $x=0$ fit etiam $z=0$, at vero sumpto $x=1$ prodit $z=\infty$,

hoc integrale a termino $z=0$ usque ad $z=\infty$ extendi debet. Notum[1]) autem est valorem hoc modo resultantem esse $\frac{\pi}{n\sin.\frac{p\pi}{n}}$.

8. Progrediamur nunc ad ipsum fundamentum, unde omnes relationes, quas quaerimus, derivari convenit et quod reductioni priori innititur; unde fit

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\frac{p+q}{p}\int\frac{x^{n+p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

ubi loco $\sqrt[n]{(1-x^n)^{n-q}}$ scribamus X, ut sit

$$\int\frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\int\frac{x^{n+p-1}\partial x}{X};$$

hinc iam simili modo, si loco p scribamus $n+p$, erit

$$\int\frac{x^{n+p-1}\partial x}{X}=\frac{n+p+q}{n+p}\int\frac{x^{2n+p-1}\partial x}{X}$$

hincque sequitur fore

$$\int\frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\cdot\frac{n+p+q}{n+p}\int\frac{x^{2n+p-1}\partial x}{X}.$$

Quodsi simili modo ulterius progrediamur, perveniemus ad hanc aequationem

$$\int\frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\cdot\frac{n+p+q}{n+p}\cdot\frac{2n+p+q}{2n+p}\int\frac{x^{3n+p-1}\partial x}{X}.$$

Quare si hoc modo in infinitum progrediamur, habebimus

$$\int\frac{x^{p-1}\partial x}{X}=\frac{p+q}{p}\cdot\frac{n+p+q}{n+p}\cdot\frac{2n+p+q}{2n+p}\cdots\frac{in+p+q}{in+p}\int\frac{x^{(i+1)n+p-1}\partial x}{X},$$

ubi i denotat numerum infinite magnum.

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1) Vide L. Euleri Commentationem 60 (indicis Enestroemiani): *De inventione integralium, si post integrationem variabili quantitati determinatus valor tribuatur*, Miscellanea Berolin. 7, 1743, p. 129, imprimis § 32; *Leonhardi Euleri Opera omnia*, series I, vol. 17, p. 35. Vide etiam L. Euleri *Institutionum calculi integralis* vol. 1, Petropoli 1768, sectio 1, cap. VIII; *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 208, imprimis p. 225. A. L.

9. Quodsi iam loco p alium quemcumque numerum r pariter ipso n minorem assumamus, erit simili modo

$$\int \frac{x^{r-1}\partial x}{X} = \frac{r+q}{r} \cdot \frac{n+r+q}{n+r} \cdot \frac{2n+r+q}{2n+r} \cdots \frac{in+r+q}{in+r} \int \frac{x^{(i+1)n+r-1}\partial x}{X},$$

ubi littera i eundem numerum infinitum designat, ita ut utrimque idem factorum numerus adsit. Dividamus iam priorem expressionem per istam, et quoniam extremae formulae integrales ob litteras p et r prae $(i+1)n$ evanescentes pro aequalibus inter se sunt habendae, facta divisione per singulos factores reperiemus hanc aequationem

$$\frac{\int x^{p-1}\partial x : X}{\int x^{r-1}\partial x : X}$$
$$= \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \cdot \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \frac{(3n+r)(3n+p+q)}{(3n+p)(3n+r+q)} \cdot \text{etc.}$$

Restituamus iam loco harum formularum integralium characteres ante [§ 7] stabilitos atque adipiscemur istam relationem notatu dignissimam

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \cdot \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \text{etc.},$$

quod productum ex infinitis membris componitur, quorum singula sunt fractiones, quarum tam numeratores quam denominatores ex binis factoribus constant. Hos factores singulos eodem numero n augeri oportet, dum a quovis membro ad sequens progredimur, unde sufficiet solum primum productum nosse, quod ergo ita repraesentabimus

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)} \cdot \text{etc.}$$

10. Quoniam litterae p et q nobis numeros quasi indefinitos significant, utamur litteris alphabeti initialibus ad numeros determinatos designandos eritque eodem modo

$$\frac{(a,b)}{(\alpha,b)} = \frac{\alpha(a+b)}{a(\alpha+b)} \cdot \frac{(n+\alpha)(n+a+b)}{(n+a)(n+\alpha+b)} \cdot \text{etc.}$$

Hic iam loco α scribamus $a+c$ et productum infinitum hanc induet formam

$$\frac{(a,b)}{(a+c,b)} = \frac{(a+c)(a+b)}{a(a+c+b)} \cdot \frac{(n+a+c)(n+a+b)}{(n+a)(n+a+c+b)} \cdot \text{etc.},$$

in quo producto ambae litterae b et c manifesto permutari possunt, unde idem productum infinitum etiam exprimet valorem huius formae $\frac{(a,c)}{(a+b,c)}$, unde sequitur ista aequalitas maxime memorabilis

$$\frac{(a,b)}{(a+c,b)}=\frac{(a,c)}{(a+b,c)};$$

fractionibus igitur sublatis habebimus istud insigne theorema

$$(a, b)\ (a+b, c)=(a, c)\ (a+c, b)$$

huicque theoremati universa analysis, qua utemur, erit superstructa.

11. Cum ob rationes supra allegatas numeri p et q exponentem n superare non debeant, etiam in forma theorematis modo allati singuli termini ibi occurrentes, qui sunt $a, b, c, a+b$ et $a+c$, quovis casu exponentem n superare non debent sicque nec $a+b$ neque $a+c$ maior capi poterit quam n. Hic autem primo observo litteras b et c inter se inaequales statui debere; si enim esset $c=b$, aequalitas in theoremate expressa foret identica; hanc ob rem perpetuo assumemus $b>c$, ita ut maximus terminus in theoremate sit $a+b$, quem ergo exponentem n quovis casu excedere non oportet, quamobrem evolutionem formae generalis in theoremate contentae ita in classes distribuamus, quae inter se per maximum valorem termini $a+b$ distinguantur. Cum igitur nulla litterarum a, b, c nihilo aequalis sumi queat ac esse debeat $b>c$, minimus valor, quem terminus $a+b$ recipere potest, erit 3, in quo ergo primam classem constituemus; sequentes vero classes constituentur, dum termino $a+b$ valores 4, 5, 6, 7 etc. tribuantur.

I. EVOLUTIO CLASSIS QUA $a+b=3$

12. Hic ergo necessario erit $a=1$, $b=2$ et $c=1$, ita ut hic nulla varietas locum inveniat, unde theorema nostrum suppeditat hanc unicam relationem

$$(1, 2)\,(3, 1)=(1, 1)\,(2, 2).$$

Dummodo igitur exponens n non fuerit minor quam 3[1]), semper haec insignis

1) Manifestum est hanc conditionem necessariam non esse. A. L.

relatio locum habet

$$\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-2}}}\cdot\int\frac{xx\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}=\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}\cdot\int\frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-2}}},$$

quae forma, quia in quolibet charactere terminos inter se permutare licet, etiam hoc modo repraesentari poterit

$$\int\frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}\cdot\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-3}}}=\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}}\cdot\int\frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-3}}}.$$

II. EVOLUTIO CLASSIS QUA $a+b=4$

13. Quoniam b binario minor esse nequit, hic erit vel $b=2$ vel $b=3$. Sit igitur primo $b=2$ eritque $a=2$ et $c=1$; unde ex nostro theoremate sequitur haec relatio

$$(2, 2)\,(4, 1)=(2, 1)\,(3, 2),$$

quae forma manifesto oritur ex classe prima, si ibi termini priores cuiusque characteris unitate augeantur; id quod etiam inde intelligere licet, quod omnes termini priores litteram a continent, qua unitate aucta processus semper fit ad classem sequentem.

14. Deinde vero hic quoque statui potest $b=3$, unde fit $a=1$; at vero littera c iam duos valores, vel 1 vel 2, sortiri poterit; priore casu, quo $c=1$, prodibit ista aequatio

$$(1, 3)\,(4, 1)=(1, 1)\,(2, 3);$$

alter vero casus, quo $c=2$, praebet hanc aequationem

$$(1, 3)\,(4, 2)=(1, 2)\,(3, 3).$$

Sicque haec classis omnino sequentes tres relationes continebit

1. $(2, 2)\,(4, 1)=(2, 1)\,(3, 2)$,
2. $(1, 3)\,(4, 1)=(1, 1)\,(2, 3)$,
3. $(1, 3)\,(4, 2)=(1, 2)\,(3, 3)$.

III. EVOLUTIO CLASSIS QUA $a+b=5$

15. In hac igitur classe primo occurrent tres relationes praecedentes, si modo termini priores cuiusque characteris unitate augeantur; hinc enim casus exsurgent, quibus est vel $b=2$ vel $b=3$. De novo igitur hic accedent casus, quibus $b=4$ et $a=1$, ubi ergo erit vel $c=1$ vel $c=2$ vel $c=3$, quibus ergo tribus casibus evolutis omnino in hac classe sex continebuntur relationes, quae erunt

1. $(3, 2)(5, 1)=(3, 1)(4, 2)$,
2. $(2, 3)(5, 1)=(2, 1)(3, 3)$,
3. $(2, 3)(5, 2)=(2, 2)(4, 3)$,
4. $(1, 4)(5, 1)=(1, 1)(2, 4)$,
5. $(1, 4)(5, 2)=(1, 2)(3, 4)$,
6. $(1, 4)(5, 3)=(1, 3)(4, 4)$.

IV. EVOLUTIO CLASSIS QUA $a+b=6$

16. Hic igitur primum occurrent omnes relationes proxime praecedentes, si modo termini priores cuiusque characteris unitate augeantur; hi scilicet nascuntur, si fuerit vel $b=2$ vel $b=3$ vel $b=4$. Praeterea vero insuper accedent casus $b=5$ et $a=1$, ubi littera c recipere poterit valores 1, 2, 3, 4, sicque omnino in hac classe occurrent decem relationes sequentes

1. $(4, 2)(6, 1)=(4, 1)(5, 2)$,
2. $(3, 3)(6, 1)=(3, 1)(4, 3)$,
3. $(3, 3)(6, 2)=(3, 2)(5, 3)$,
4. $(2, 4)(6, 1)=(2, 1)(3, 4)$,
5. $(2, 4)(6, 2)=(2, 2)(4, 4)$,
6. $(2, 4)(6, 3)=(2, 3)(5, 4)$,
7. $(1, 5)(6, 1)=(1, 1)(2, 5)$,
8. $(1, 5)(6, 2)=(1, 2)(3, 5)$,
9. $(1, 5)(6, 3)=(1, 3)(4, 5)$,
10. $(1, 5)(6, 4)=(1, 4)(5, 5)$.

V. EVOLUTIO CLASSIS QUA $a+b=7$

17. Hic igitur primo occurrent omnes relationes classis IV, postquam scilicet omnes terminos priores singulorum characterum unitate auxerimus, quos igitur hic apposuisse non erit necesse, ac sufficiet eas tantum relationes hic exponere, quae de novo accedunt et ex valore $b=6$ oriuntur existente $a=1$; ubi pro c sumi poterunt numeri 1, 2, 3, 4, 5, ita ut harum numerus sit quinque. Hae ergo relationes sunt

1. $(1, 6)\,(7, 1) = (1, 1)\,(2, 6)$,
2. $(1, 6)\,(7, 2) = (1, 2)\,(3, 6)$,
3. $(1, 6)\,(7, 3) = (1, 3)\,(4, 6)$,
4. $(1, 6)\,(7, 4) = (1, 4)\,(5, 6)$,
5. $(1, 6)\,(7, 5) = (1, 5)\,(6, 6)$.

VI. EVOLUTIO CLASSIS QUA $a+b=8$

18. In hac iam classe primo occurrent omnes decem relationes classis IV, dum scilicet omnes termini priores binario augentur; praeterea quoque accedent quinque relationes in classe V allatae, dum partes priores unitate augebuntur; praeter has vero de novo accedent sex sequentes relationes ex valoribus $a=1$ et $b=7$ oriundae, dum litterae c valores 1, 2, 3, 4, 5, 6 ordine tribuuntur, quae ergo erunt

1. $(1, 7)\,(8, 1) = (1, 1)\,(2, 7)$,
2. $(1, 7)\,(8, 2) = (1, 2)\,(3, 7)$,
3. $(1, 7)\,(8, 3) = (1, 3)\,(4, 7)$,
4. $(1, 7)\,(8, 4) = (1, 4)\,(5, 7)$,
5. $(1, 7)\,(8, 5) = (1, 5)\,(6, 7)$,
6. $(1, 7)\,(8, 6) = (1, 6)\,(7, 7)$.

VII. EVOLUTIO CLASSIS QUA $a+b=9$

19. Ut omnes relationes ad hanc classem pertinentes adipiscamur, notandum est primo hic occurrere decem relationes classis IV, dum partes priores ternario augentur. Secundo adiici oportet quinque relationes in classe V exhibitas, ubi partes priores binario augeri debent. Tertio huc referri debent sex relationes classis VI partes priores unitate augendo. Insuper vero de novo accedent septem relationes ex valoribus $a=1$ et $b=8$ natae, dum litterae c tribuuntur ordine valores 1, 2, 3, 4, 5, 6, 7. Hae relationes sunt

$$\begin{aligned}
&1.\ (1, 8)\,(9, 1) = (1, 1)\,(2, 8),\\
&2.\ (1, 8)\,(9, 2) = (1, 2)\,(3, 8),\\
&3.\ (1, 8)\,(9, 3) = (1, 3)\,(4, 8),\\
&4.\ (1, 8)\,(9, 4) = (1, 4)\,(5, 8),\\
&5.\ (1, 8)\,(9, 5) = (1, 5)\,(6, 8),\\
&6.\ (1, 8)\,(9, 6) = (1, 6)\,(7, 8),\\
&7.\ (1, 8)\,(9, 7) = (1, 7)\,(8, 8).
\end{aligned}$$

20. Hinc iam ordo progressionis tam clare perspicitur, ut superfluum foret has evolutiones ulterius prosequi; quandoquidem ob ingentem multitudinem relationum, quae in sequentibus classibus occurrerent, nimis molestum foret omnes percurrere. Quin etiam nostrum institutum vix permittere videtur, ut in nostra formula generali exponentem n ultra sex vel septem augeamus, siquidem omnes relationes ad eum pertinentes enumerare voluerimus. Sin autem animus sit aliquas tantum expendere, classes allatae abunde sufficiunt, dum termini priores cuiusque classis quovis numero augebuntur.

21. His iam classibus expeditis formulam integralem propositam

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

secundum diversos valores exponentis n pertractemus, dum scilicet successive assumemus $n=3$, $n=4$, $n=5$ etc., et pro quolibet ordine omnes relationes, quae in eo occurrere possunt, expendamus. Evidens autem est, quicumque numerus exponenti n tribuatur, formulas omnium classium inferiorum, in

quibus scilicet terminus $a+b$ non superet n, in usum vocari posse. Ex quo intelligitur, si fuerit $n=3$, unicam relationem locum invenire; statim autem ac n magis augetur, numerus omnium relationum mox ita increscit, ut nimis molestum foret omnes recensere. Hos igitur diversos ordines ex exponente n constituendos a primo incipiendo ordine evolvamus.

ORDO I QUO $n=3$ ET FORMULA

$$(p, q)=\int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^{3-q}}}=\int \frac{x^{q-1}\partial x}{\sqrt[3]{(1-x^3)^{3-p}}}$$

22. Cum hic sit $n=3$, erit [§ 7]

$$(3, 1)=1;$$

formulae autem integrales huius ordinis erunt tres, scilicet

$$1.\ (1, 1),\quad 2.\ (1, 2),\quad 3.\ (2, 2),$$

quarum media ob $1+2=3$ a circulo pendet; quae ergo quia est cognita, ponatur

$$(1, 2)=\frac{\pi}{3 \sin. \frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}=A.$$

Hic igitur tantum classis prima locum habet, quae nobis hanc unicam aequationem suppeditat

$$A=(1, 1)(2, 2).$$

23. Hinc ergo patet productum ex binis formulis transcendentibus $(1, 1)$ et $(2, 2)$ aequari quantitati circulari $A=\frac{2\pi}{3\sqrt{3}}$, ita ut pro ipsis formulis integralibus habeamus hanc relationem

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}}=\frac{2\pi}{3\sqrt{3}};$$

unde si altera harum duarum formularum fuerit cognita, etiam valor alterius assignari potest. Spectemus ergo priorem, quasi nobis esset cognita, etiamsi

sit transcendens, eamque ponamus

$$(1, 1) = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = P$$

eritque

$$(2, 2) = \frac{A}{P}.$$

Sicque nihil praeterea in hoc ordine notandum relinquitur.

ORDO II QUO $n = 4$ ET FORMULA

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[4]{(1-x^4)^{4-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[4]{(1-x^4)^{4-p}}}$$

24. Cum igitur hic sit $n = 4$, erit

$$(4, 1) = 1 \quad \text{et} \quad (4, 2) = \tfrac{1}{2};$$

formulae autem integrales ad hunc ordinem pertinentes erunt sex sequentes

$$1.\ (1, 1),\quad 2.\ (1, 2),\quad 3.\ (1, 3),\quad 4.\ (2, 2),\quad 5.\ (2, 3),\quad 6.\ (3, 3),$$

inter quas ergo reperiuntur duae formulae circulares $(1, 3)$ et $(2, 2)$, quas propterea litteris A et B designemus ponendo

$$(1, 3) = \frac{\pi}{4\sin.\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} = A$$

et

$$(2, 2) = \frac{\pi}{4\sin.\frac{2\pi}{4}} = \frac{\pi}{4} = B,$$

ita ut sit

$$\frac{A}{B} = \sqrt{2}.$$

25. In hoc ergo ordine aequationes tam primae quam secundae classis locum habere possunt; secunda autem classis nobis has tres praebet aequationes

$$1.\ B = (2, 1)\,(3, 2),\quad 2.\ A = (1, 1)\,(2, 3),\quad 3.\ A = 2\,(1, 2)\,(3, 3),$$

classis vero prima insuper dat hanc aequationem

$$A(1, 2) = (1, 1)B$$

sive $\frac{A}{B} = \frac{(1,1)}{(1,2)}$, quae autem aequatio iam ex duabus prioribus deducitur; namque ob $(3, 2) = (2, 3)$ secunda per primam divisa dabit $\frac{A}{B} = \frac{(1,1)}{(1,2)} = \sqrt{2}$, ita ut ratio inter has duas formulas sit algebraica, quae ergo imprimis notari meretur

$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} : \int \frac{\partial x}{\sqrt{(1-x^4)}} = \sqrt{2}.$$

26. Iam in hoc ordine praeter binas formulas circulares $(1, 3) = A$ et $(2, 2) = B$ tamquam cognitam etiam introducamus formulam $(1, 2)$, quae in ordine praecedente erat circularis, nunc autem est transcendens, eamque ponamus

$$(1, 2) = \int \frac{\partial x}{\sqrt{(1-x^4)}} = P;$$

ubi caveatur, ne litterae A et P cum iis confundantur, quibus in formulis praecedentibus sumus usi, id quod etiam de ordinibus sequentibus est tenendum. His igitur litteris introductis aequationes nostrae erunt sequentes tres

$$1.\ B = P(3, 2), \quad 2.\ A = (1, 1)(2, 3), \quad 3.\ A = 2P(3, 3),$$

quandoquidem vidimus quartam in praecedentibus iam contineri.

27. Ope harum trium aequationum ergo ternas formulas integrales etiamnunc incognitas per ternas A, B et P, quas ut datas spectamus, determinare licebit. Ex prima enim fit $(3, 2) = \frac{B}{P}$; ex tertia autem fit $(3, 3) = \frac{A}{2P}$; tum vero ex secunda colligitur $(1, 1) = \frac{A}{(3,2)} = \frac{AP}{B}$. Cum igitur in hoc ordine omnino sint sex formulae integrales, earum ternae per tres reliquas definiri possunt, quas determinationes igitur ob oculos posuisse iuvabit:

$$1.\ (1, 3) = A = \frac{\pi}{2\sqrt{2}}, \qquad 4.\ (1, 1) = \frac{AP}{B},$$

$$2.\ (2, 2) = B = \frac{\pi}{4}, \qquad 5.\ (2, 3) = \frac{B}{P},$$

$$3.\ (1, 2) = P = \int \frac{\partial x}{\sqrt{(1-x^4)}}, \qquad 6.\ (3, 3) = \frac{A}{2P}.$$

Ex postremis ergo erit

$$(2, 3):(3, 3) = 2B : A = \sqrt{2} : 1,$$

ita ut etiam hae duae formulae inter se habeant rationem algebraicam, qua est

$$\int \frac{xx\partial x}{\sqrt{(1-x^4)}} = \sqrt{2}\int \frac{xx\partial x}{\sqrt[4]{(1-x^4)}}.$$

Aliis insignibus relationibus utpote satis cognitis hic non immoramur.

ORDO III QUO $n=5$ ET FORMULA

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[5]{(1-x^5)^{5-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[5]{(1-x^5)^{5-p}}}$$

28. Hic igitur ob $n=5$ ante omnia erit

$$(5, 1) = 1, \quad (5, 2) = \tfrac{1}{2}, \quad (5, 3) = \tfrac{1}{3};$$

formulae autem integrales huius ordinis erunt hae decem

1. (1, 1), 2. (1, 2), 3. (1, 3), 4. (1, 4), 5. (2, 2),
6. (2, 3), 7. (2, 4), 8. (3, 3), 9. (3, 4), 10. (4, 4),

inter quas quarta et sexta sunt circulares, quas ergo ita designemus

$$(1, 4) = \frac{\pi}{5\sin.\frac{1}{5}\pi} = A$$

et

$$(2, 3) = \frac{\pi}{5\sin.\frac{2\pi}{5}} = B.$$

Praeterea vero binas formulas, quae in ordine praecedenti erant circulares, nunc autem sunt transcendentes, etiam peculiaribus litteris notemus, scilicet

$$(1, 3) = P \quad \text{et} \quad (2, 2) = Q.$$

Mox enim patebit, dummodo etiam istae formulae tamquam cognitae spectentur, reliquas sex omnes per has quatuor determinari posse.

29. Quoniam hic tres classes priores locum habere possunt, consideremus primo aequationes, quas tertia classis suppeditat et quae introductis his valoribus erunt

$$
\begin{array}{ll}
1.\ B = P(4, 2), & 4.\ A = (1, 1)\,(2, 4), \\
2.\ B = (2, 1)\,(3, 3), & 5.\ A = 2\,(1, 2)\,(3, 4), \\
3.\ B = 2\,Q(4, 3), & 6.\ A = 3\,P(4, 4).
\end{array}
$$

Quas hoc modo succinctius repraesentare licet

$$A = (1, 1)\,(2, 4) = 2\,(1, 2)\,(3, 4) = 3\,P(4, 4),$$

$$B = P(4, 2) = (2, 1)\,(3, 3) = 2\,Q(4, 3),$$

ubi sex occurrunt producta ex binis formulis integralibus, quae singula quantitati circulari aequantur, unde totidem egregia theoremata formari possent, nisi hinc iam clare in oculos incurrerent.

30. Iam videamus, quot formulas integrales incognitas ex quatuor cognitis A, B, P et Q definire queamus; at vero prima dat $(4, 2) = \frac{B}{P}$, tertia praebet $(4, 3) = \frac{B}{2Q}$, sexta dat $(4, 4) = \frac{A}{3P}$; hinc autem porro ex quarta deducimus $(1, 1) = \frac{A}{(2, 4)} = \frac{AP}{B}$, ex quinta vero deducimus $(1, 2) = \frac{A}{2(3, 4)} = \frac{AQ}{B}$. Denique ex secunda elicimus $(3, 3) = \frac{B}{(2, 1)} = \frac{BB}{AQ}$ sicque ex his sex aequationibus sex determinationes sumus adepti; atque adeo per litteras A, B, P et Q valores omnium reliquarum litterarum assignavimus.

31. Quoniam igitur hactenus tantum classe tertia sumus usi, consideremus etiam aequationes secundae classis, quae sunt

$$1.\ AQ = B(2, 1),$$

$$2.\ AP = B(1, 1)$$

et

$$3.\ P(4, 2) = (1, 2)\,(3, 3);$$

verum si hic valores modo inventos substituamus, aequationes mere identicae resultant, ita ut hinc nulla nova determinatio sequatur. Idem usu venit ex

aequatione primae classis, quae erat $(2, 1)(3, 1)=(1, 1)(2, 2)$, quae facta substitutione quoque fit identica, ita ut duae priores classes nihil novi involvant. Neque tamen hinc concludere licet etiam in sequentibus ordinibus classes praecedentes praetermitti posse, siquidem in ordine sequente statim contrarium se manifestabit.

32. Cum igitur hic ordo complectatur decem formulas integrales, earum valores per quatuor litteras A, B, P et Q ordine ita aspectui exponamus:

1. $(1, 1)=\frac{AP}{B}$,	6. $(2, 3)=B$,
2. $(1, 2)=\frac{AQ}{B}$,	7. $(2, 4)=\frac{B}{P}$,
3. $(1, 3)=P$,	8. $(3, 3)=\frac{BB}{AQ}$,
4. $(1, 4)=A$,	9. $(3, 4)=\frac{B}{2Q}$,
5. $(2, 2)=Q$,	10. $(4, 4)=\frac{A}{3P}$.

33. Cum sit

$$\frac{A}{B}=\frac{\sin.\frac{2}{5}\pi}{\sin.\frac{1}{5}\pi}=2\cos.\frac{1}{5}\pi,$$

tum vero

$$\cos.\frac{1}{5}\pi=\frac{1+\sqrt{5}}{4},$$

erit

$$\frac{A}{B}=\frac{1+\sqrt{5}}{2}$$

ideoque quantitas algebraica. Hinc igitur aliquot paria formularum integralium exhiberi poterunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{(1, 1)}{(1, 3)}=\frac{1+\sqrt{5}}{2}, \quad \frac{(1, 2)}{(2, 2)}=\frac{1+\sqrt{5}}{2}, \quad \frac{(3, 4)}{(3, 3)}=\frac{1+\sqrt{5}}{4}, \quad \frac{(4, 4)}{(2, 4)}=\frac{1+\sqrt{5}}{6},$$

unde totidem egregia theoremata condi possent, nisi ex his formulis manifesto elucerent.

ORDO IV QUO $n=6$ ET FORMULA

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[6]{(1-x^6)^{6-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[6]{(1-x^6)^{6-p}}}$$

34. Quoniam hic est $n=6$, habebimus ante omnia

$$(6, 1) = 1, \quad (6, 2) = \frac{1}{2}, \quad (6, 3) = \frac{1}{3}, \quad (6, 4) = \frac{1}{4};$$

formularum autem integralium in hoc ordine occurrentium numerus est quindecim, quae sunt

1. $(1, 1)$, 2. $(1, 2)$, 3. $(1, 3)$, 4. $(1, 4)$, 5. $(1, 5)$,
6. $(2, 2)$, 7. $(2, 3)$, 8. $(2, 4)$, 9. $(2, 5)$, 10. $(3, 3)$,
11. $(3, 4)$, 12. $(3, 5)$, 13. $(4, 4)$, 14. $(4, 5)$, 15. $(5, 5)$,

inter quas reperiuntur tres circulares, quas singulari modo designemus, scilicet

$$(1, 5) = \frac{\pi}{6 \sin. \frac{1}{6}\pi} = \frac{\pi}{3} = A,$$

$$(2, 4) = \frac{\pi}{6 \sin. \frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = B$$

et

$$(3, 3) = \frac{\pi}{6 \sin. \frac{3\pi}{6}} = \frac{\pi}{6} = C,$$

ita ut sit

$$A = 2C.$$

Praeterea vero ambas formulas, quae in ordine praecedente erant circulares, nunc vero sunt transcendentes, statuamus

$$(1, 4) = P \quad \text{et} \quad (2, 3) = Q.$$

His factis denominationibus evolvamus decem aequationes classis quartae, quae sunt

1. $B=P(5,2)$,	6. $B=3\,Q(5,4)$,
2. $C=(3,1)\,(4,3)$,	7. $A=(1,1)\,(5,2)$,
3. $C=2\,Q(5,3)$,	8. $A=2(1,2)\,(3,5)$,
4. $B=(2,1)\,(3,4)$,	9. $A=3(1,3)\,(4,5)$,
5. $B=2(2,2)\,(4,4)$,	10. $A=4\,P(5,5)$,

quas ita succinctius referre licet

$$A=(1,1)\,(5,2)=2(1,2)\,(3,5)=3(1,3)\,(4,5)=4\,P(5,5),$$

$$B=P(5,2)=(2,1)\,(3,4)=2(2,2)\,(4,4)=3\,Q(4,5),$$

$$C=(3,1)\,(4,3)=2\,Q(5,3).$$ [1]

Ecce ergo decem producta ex binis formulis integralibus, quorum singula quantitati circulari aequantur.

35. Cum deinde sit $\frac{A}{B}=\sqrt{3}$ et $\frac{A}{C}=2$, tum vero etiam $\frac{B}{C}=\frac{2}{\sqrt{3}}$, plura paria binarum formularum integralium exhiberi possunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{A}{B}=\sqrt{3}=\frac{(1,1)}{(1,4)}=\frac{2\,(3,5)}{(3,4)}=\frac{(1,3)}{(2,3)}=\frac{4\,(5,5)}{(5,2)},$$

$$\frac{A}{C}=\;2\;=\frac{(1,2)}{(2,3)}=\frac{3\,(4,5)}{(4,3)},$$

$$\frac{B}{C}=\frac{2}{\sqrt{3}}=\frac{(1,2)}{(1,3)}=\frac{3\,(4,5)}{2\,(3,5)}.$$ [2]

1) Editio princeps:

$$C=(3,1)\,(5,2)=2\,Q\,(5,3),$$

unde errores in formulis sequentibus nati sunt. Correxit A. L.

2) Editio princeps:

$$\frac{A}{C}=\;2\;=\frac{(1,1)}{(1,3)}=\frac{(1,2)}{(2,3)}=\frac{3\,(4,5)}{(2,5)},$$

$$\frac{B}{C}=\frac{2}{\sqrt{3}}=\frac{(1,4)}{(1,3)}=\frac{3\,(4,5)}{2\,(3,5)}.$$

Correxit A. L.

36. Quodsi iam quinque formulas litteris A, B, C, P et Q designatas tamquam cognitas spectemus, videamus, quomodo reliquae formulae per eas definiri queant. Ac primo quidem percurramus decem aequationes classis quartae supra allatas, quarum prima dabit $(5, 2) = \frac{B}{P}$, tertia dat $(5, 3) = \frac{C}{2Q}$, sexta praebet $(5, 4) = \frac{B}{3Q}$, decima dat $(5, 5) = \frac{A}{4P}$. Quodsi iam hos valores in reliquis surrogemus, septima praebet $(1, 1) = \frac{A}{(5, 2)} = \frac{AP}{B}$, octava dat $(1, 2) = \frac{A}{2(3, 5)} = \frac{AQ}{C}$, nona dat $(3, 1) = \frac{A}{3(4, 5)} = \frac{AQ}{B}$. Porro vero quarta dat $(3, 4) = \frac{B}{(2, 1)} = \frac{BC}{AQ}$, quem valorem etiam secunda praebet.[1]) At vero ex aequatione quinta nullum valorem elicere possumus, quia neque formula $(2, 2)$ nec $(4, 4)$ etiamnunc constat. Causa est, quia duae reliquarum aequationum eandem determinationem produxerunt.

37. Coacti igitur sumus ad aequationes praecedentium classium confugere atque adeo ex prima classe

$$(1, 2)(3, 1) = (1, 1)(2, 2)$$

statim colligimus

$$(2, 2) = \frac{(1, 2)(3, 1)}{(1, 1)} = \frac{AQQ}{CP},$$

qui valor in quinta aequatione substitutus suppeditat postremam aequationem, nempe

$$(4, 4) = \frac{B}{2(2, 2)} = \frac{BCP}{2AQQ}.$$

Omnes igitur hos valores hic ordine referemus:

1. $(1, 1) = \frac{AP}{B}$,	4. $(1, 4) = P$,	7. $(2, 3) = Q$,
2. $(1, 2) = \frac{AQ}{C}$,	5. $(1, 5) = A$,	8. $(2, 4) = B$,
3. $(1, 3) = \frac{AQ}{B}$,	6. $(2, 2) = \frac{AQQ}{CP}$,	9. $(2, 5) = \frac{B}{P}$,

1) Editio princeps: *Quodsi iam hos valores in reliquis surrogemus, secunda dabit* $(3, 1) = \frac{C}{(4, 3)} = \frac{AQ}{B}$, *septima praebet . . . nona dat* $(3, 1) = \frac{A}{3(4, 5)} = \frac{AQ}{B}$, *quem valorem etiam secunda praebuit. Porro vero quarta dat* $(3, 4) = \frac{B}{(2, 1)} = \frac{BC}{AQ}$. Correxit A. L.

$$10.\ (3,3)=C, \qquad 12.\ (3,5)=\frac{C}{2Q}, \qquad 14.\ (4,5)=\frac{B}{3Q},$$

$$11.\ (3,4)=\frac{BC}{AQ}, \qquad 13.\ (4,4)=\frac{BCP}{2AQQ}, \qquad 15.\ (5,5)=\frac{A}{4P}.$$

38. Cum autem in hoc ordine etiam aequationes tam classis secundae quam tertiae valere debeant, videamus, utrum valores inventi his classibus conveniant an vero forte novam determinationem suppeditent? Facta autem substitutione in tribus aequationibus secundae classis ad identitatem pervenitur, quod idem quoque in aequationibus tertiae classis contingere debet, id quod evolventi mox patebit. Unde memorabile est omnes aequationes in quatuor primis classibus contentas, quarum numerus est 20, tantum decem determinationes in se complecti.

ORDO V QUO $n=7$ ET FORMULA

$$(p,q)=\int\frac{x^{p-1}\partial x}{\sqrt[7]{(1-x^7)^{7-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[7]{(1-x^7)^{7-p}}}$$

39. Quia hic $n=7$, ante omnia habebimus valores absolutos

$$(7,1)=1, \quad (7,2)=\tfrac{1}{2}, \quad (7,3)=\tfrac{1}{3}, \quad (7,4)=\tfrac{1}{4} \quad \text{et} \quad (7,5)=\tfrac{1}{5},$$

deinde inter formulas integrales huius ordinis imprimis notari debent circulares, quas hoc modo designemus:

$$(1,6)=\frac{\pi}{7\sin.\frac{\pi}{7}}=A,$$

$$(2,5)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=B,$$

$$(3,4)=\frac{\pi}{7\sin.\frac{3\pi}{7}}=C.$$

Praeterea vero peculiaribus litteris notentur eae formulae, quae in ordine praecedenti erant circulares, hic autem valores transcendentes sortiuntur, qui sint

$$(1, 5) = P, \quad (2, 4) = Q \quad \text{et} \quad (3, 3) = R;$$

per has enim sex litteras videbimus omnes reliquas formulas huius ordinis determinari posse.

40. Quoniam supra non omnes aequationes quintae classis expressimus, eas hic coniunctim exhibeamus et ad nostrum casum accommodemus:

I.	$(1, 6)(7, 1) = (1, 1)(2, 6)$	$A = \ (1, 1)(2, 6)$,
II.	$(1, 6)(7, 2) = (1, 2)(3, 6)$	$A = 2(1, 2)(3, 6)$,
III.	$(1, 6)(7, 3) = (1, 3)(4, 6)$	$A = 3(1, 3)(4, 6)$,
IV.	$(1, 6)(7, 4) = (1, 4)(5, 6)$	$A = 4(1, 4)(5, 6)$,
V.	$(1, 6)(7, 5) = (1, 5)(6, 6)$	$A = 5\ P\ (6, 6)$,
VI.	$(2, 5)(7, 1) = (2, 1)(3, 5)$	$B = \ (2, 1)(3, 5)$,
VII.	$(2, 5)(7, 2) = (2, 2)(4, 5)$	$B = 2(2, 2)(4, 5)$,
VIII.	$(2, 5)(7, 3) = (2, 3)(5, 5)$	$B = 3(2, 3)(5, 5)$,
IX.	$(2, 5)(7, 4) = (2, 4)(6, 5)$	$B = 4\ Q\ (6, 5)$,
X.	$(3, 4)(7, 1) = (3, 1)(4, 4)$	$C = \ (3, 1)(4, 4)$,
XI.	$(3, 4)(7, 2) = (3, 2)(5, 4)$	$C = 2(3, 2)(5, 4)$,
XII.	$(3, 4)(7, 3) = (3, 3)(6, 4)$	$C = 3\ R\ (6, 4)$,
XIII.	$(4, 3)(7, 1) = (4, 1)(5, 3)$	$C = \ (4, 1)(5, 3)$,
XIV.	$(4, 3)(7, 2) = (4, 2)(6, 3)$	$C = 2\ Q\ (6, 3)$,
XV.	$(5, 2)(7, 1) = (5, 1)(6, 2)$	$B = \ P\ (6, 2)$.

Hic igitur habemus quina producta formulae A aequalia totidemque formulis B et C aequalia.

41. Omnino autem in hoc ordine occurrunt 21 formulae integrales, ex quibus sex litteris A, B, C, P, Q et R designavimus, per quas igitur reliquas quindecim formulas integrales definiri oportet, quae sunt

1. $(1, 1)$, 2. $(1, 2)$, 3. $(1, 3)$, 4. $(2, 2)$, 5. $(1, 4)$,
6. $(2, 3)$, 7. $(2, 6)$, 8. $(3, 5)$, 9. $(4, 4)$, 10. $(3, 6)$,
11. $(4, 5)$, 12. $(4, 6)$, 13. $(5, 5)$, 14. $(5, 6)$, 15. $(6, 6)$.

42. Videamus igitur, quot harum formularum ex superioribus quindecim aequationibus determinare liceat; ac primo quidem ex aequationibus V, IX, XII, XIV et XV immediate deducuntur sequentes formulae

$$(6,6)=\frac{A}{5P},\quad (6,5)=\frac{B}{4Q},\quad (6,4)=\frac{C}{3R},\quad (6,3)=\frac{C}{2Q},\quad (6,2)=\frac{B}{P}.$$

His iam inventis ex aequationibus I, II, III et IV derivamus has formulas

$$(1,1)=\frac{AP}{B},\quad (1,2)=\frac{AQ}{C},\quad (1,3)=\frac{AR}{C},\quad (1,4)=\frac{AQ}{B}.$$

Ex his vero valoribus per aequationes VI, X et XIII colligimus

$$(3,5)=\frac{BC}{AQ},\quad (4,4)=\frac{CC}{AR}\quad \text{et}\quad (5,3)=\frac{BC}{AQ},$$

ubi notasse iuvabit eundem valorem pro (5, 3) prodiisse ex aequationibus VI et XIII. Ex reliquis autem aequationibus VII, VIII et XI nihil concludere licet, unde istae quatuor formulae (2, 2), (2, 3), (5, 4) et (5, 5) nobis etiamnunc manent incognitae.

43. Recurrere ergo coacti sumus ad aequationes praecedentium classium, quippe quae aeque ad nostrum ordinem pertinent atque aequationes classis quintae; hanc ob rem simili modo aequationes classis quartae hic apponamus et ad nostrum casum applicemus:

$$\begin{array}{rl|l}
\text{I.} & (1,5)(6,1)=(1,1)(2,5) & PA \quad =(1,1)B, \\
\text{II.} & (1,5)(6,2)=(1,2)(3,5) & P(6,2)=(1,2)(3,5), \\
\text{III.} & (1,5)(6,3)=(1,3)(4,5) & P(6,3)=(1,3)(4,5), \\
\text{IV.} & (1,5)(6,4)=(1,4)(5,5) & P(6,4)=(1,4)(5,5), \\
\text{V.} & (2,4)(6,1)=(2,1)(3,4) & QA \quad =(2,1)C, \\
\text{VI.} & (2,4)(6,2)=(2,2)(4,4) & Q(6,2)=(2,2)(4,4), \\
\text{VII.} & (2,4)(6,3)=(2,3)(5,4) & Q(6,3)=(2,3)(5,4), \\
\text{VIII.} & (3,3)(6,1)=(3,1)(4,3) & RA \quad =(3,1)C, \\
\text{IX.} & (3,3)(6,2)=(3,2)(5,3) & R(6,2)=(3,2)(5,3), \\
\text{X.} & (4,2)(6,1)=(4,1)(5,2) & QA \quad =(4,1)B.
\end{array}$$

44. Ex aequationibus I, V, VIII et X immediate concludimus has formulas

$$(1,1)=\frac{PA}{B},\quad (2,1)=\frac{QA}{C},\quad (3,1)=\frac{AR}{C},\quad (4,1)=\frac{AQ}{B},$$

quos autem valores iam ante adepti sumus. Secunda aequatio, si formulae iam inventae substituantur, praebet aequationem identicam. Ex tertia autem poterimus definire formulam (4, 5), cuius valor hinc colligitur

$$(4,5)=\frac{CCP}{2AQR}.$$

Simili modo ex quarta elicimus

$$(5,5)=\frac{BCP}{3AQR}.$$

Porro ex aequatione sexta concludimus fore

$$(2,2)=\frac{ABQR}{CCP}.$$

Deinde septima aequatio dat

$$(2,3)=\frac{AQR}{CP}.$$

Nona vero aequatio etiam praebet $(3,2)=\frac{AQR}{CP}$. Sicque omnes quindecim formulas incognitas determinavimus per sex litteras cognitas A, B, C, P, Q et R.

45. Valores igitur omnium formularum huius ordinis hic aspectui coniunctim exponamus:

$(1,6)=A$	$(6,2)=\frac{B}{P}$	$(1,1)=\frac{AP}{B}$	$(3,5)=\frac{BC}{AQ}$	$(2,3)=\frac{AQR}{CP}$,
$(2,5)=B$	$(6,3)=\frac{C}{2Q}$	$(1,2)=\frac{AQ}{C}$	$(4,4)=\frac{CC}{AR}$	$(4,5)=\frac{CCP}{2AQR}$,
$(3,4)=C$	$(6,4)=\frac{C}{3R}$	$(1,3)=\frac{AR}{C}$		$(5,5)=\frac{BCP}{3AQR}$,
$(1,5)=P$	$(6,5)=\frac{B}{4Q}$	$(1,4)=\frac{AQ}{B}$		$(2,2)=\frac{ABQR}{CCP}$.
$(2,4)=Q$	$(6,6)=\frac{A}{5P}$			
$(3,3)=R$				

46. Quoniam autem aequationes primae, secundae ac tertiae classis etiam in hoc ordine valent, si in iis valores hic inventos substituamus, perpetuo in aequationes identicas incidemus. Ita, cum aequatio primae classis sit

$$(1,2)(3,1)=(1,1)(2,2),$$

facta substitutione reperietur

$$(1,2)(3,1)=\frac{AAQR}{CC};$$

at vero $(1,1)(2,2)$ fit $=\frac{AAQR}{CC}$ haecque identitas etiam deprehendetur in tribus aequationibus secundae classis atque etiam in sex aequationibus tertiae classis, quemadmodum calculum instituenti mox patebit.

47. Simili modo haud difficile erit hanc investigationem ad ordines superiores extendere, neque tamen legem observare licet, secundum quam determinationes singularum formularum cuiusque ordinis progrediuntur. Interim tamen observasse iuvabit in ordine sequente sexto, ubi $n=8$ et formulae occurrunt 28, eas omnes primo per quatuor formulas circulares

$$(1,7)=A,\quad (2,6)=B,\quad (3,5)=C,\quad (4,4)=D,$$

praeterea vero per has tres transcendentes

$$(1,6)=P,\quad (2,5)=Q \quad \text{et} \quad (3,4)=R$$

determinari posse. Cum igitur quovis ordine determinatio singularum formularum praeter formulas circulares, quae utique pro cognitis haberi possunt, etiam aliquot formulas transcendentes postulat, si saltem valores harum formularum vero proxime cognoscere voluerimus, methodus adhuc desideratur istos valores proxime, veluti in fractionibus decimalibus, definiendi. Talem igitur methodum hic coronidis loco subiungemus.

PROBLEMA

48. *Proposita formula integrali cuiusque ordinis*

$$S=\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

a termino $x=0$ *usque ad* $x=1$ *extendenda investigare seriem convergentem, quae istum valorem* S *exprimat.*

SOLUTIO

Cum sit

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = (1-x^n)^{-\frac{n-q}{n}},$$

facta evolutione huius potestatis binomii more solito reperietur

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = 1 + \frac{n-q}{n}x^n + \frac{n-q}{n}\cdot\frac{2n-q}{2n}x^{2n} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}x^{3n} + \text{etc.}$$

Si haec series ducatur in $x^{p-1}\partial x$ et integretur, prodibit

$$S = \frac{x^p}{p} + \frac{n-q}{n}\cdot\frac{x^{n+p}}{n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{x^{2n+p}}{2n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{x^{3n+p}}{3n+p} + \text{etc.},$$

quae series iam evanescit posito $x = 0$; unde si ponamus $x = 1$, valor quaesitus nostrae formulae fiet

$$S = \frac{1}{p} + \frac{n-q}{n}\cdot\frac{1}{n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{n}\cdot\frac{1}{2n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{1}{3n+p} + \text{etc.}$$

49. Verum ista series, quicumque numeri pro litteris n, p et q accipiantur, nimis lente convergit, quam ut ex ea valores ipsius S saltem ad tres quatuorve figuras decimales satis exacte definiri queant; quamobrem aliam evolutionem institui conveniet, dum scilicet valorem quaesitum in duas partes resolvemus. Statuamus igitur

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}\begin{bmatrix}\text{ab } x = 0\\ \text{ad } x^n = \frac{1}{2}\end{bmatrix} = P$$

et

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}\begin{bmatrix}\text{ab } x^n = \frac{1}{2}\\ \text{ad } x = 1\end{bmatrix} = Q$$

atque evidens est fore

$$S = P + Q.$$

Nunc autem tam pro P quam pro Q haud difficulter series satis convergentes exhiberi poterunt.

50. Quod primum ad valorem P attinet, eum ex valore generali, quem supra pro S invenimus, facile derivabimus ponendo $x^n=\frac{1}{2}$, ita ut sit $x=\sqrt[n]{\frac{1}{2}}$ et $x^p=\frac{1}{\sqrt[n]{2^p}}$, quo facto pro P obtinebimus hanc seriem

$$P=\frac{1}{\sqrt[n]{2^p}}\left\{\begin{aligned}&\frac{1}{p}+\frac{n-q}{2n}\cdot\frac{1}{n+p}+\frac{n-q}{2n}\cdot\frac{2n-q}{4n}\cdot\frac{1}{2n+p}\\&+\frac{n-q}{2n}\cdot\frac{2n-q}{4n}\cdot\frac{3n-q}{6n}\cdot\frac{1}{3n+p}+\text{etc.}\end{aligned}\right\}.$$

In qua serie singuli termini plus quam in ratione dupla decrescunt, ita ut verbi gratia terminus decimus iam multo minor futurus sit quam $\frac{1}{1024}$, unde, si ad partes millionesimas certi esse velimus, sufficeret calculum ne quidem ad vicesimum usque terminum extendere.

51. Cum deinde posuerimus

$$Q=\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}\begin{bmatrix}\text{ab } x^n=\frac{1}{2}\\\text{ad } x\;=1\end{bmatrix},$$

statuamus $1-x^n=y^n$, ut sit

$$Q=\int\frac{x^{p-1}\partial x}{y^{n-q}};$$

tum vero erit $x^n=1-y^n$ ideoque $x^p=\sqrt[n]{(1-y^n)^p}$, unde differentiando colligitur

$$x^{p-1}\partial x=-y^{n-1}\partial y(1-y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$Q=-\int y^{q-1}\partial y(1-y^n)^{\frac{p-n}{n}}\begin{bmatrix}\text{ab } y^n=\frac{1}{2}\\\text{ad } y\;=0\end{bmatrix}.$$

Quando enim fit $x^n=\frac{1}{2}$, tum etiam erit $y^n=\frac{1}{2}$, at facto $x=1$ manifesto fit $y=0$; quare si terminos integrationis permutemus, etiam signum ipsius formulae immutari debet sicque fiet

$$Q=\int y^{q-1}\partial y(1-y^n)^{\frac{p-n}{n}}\begin{bmatrix}\text{ab } y\;=0\\\text{ad } y^n=\frac{1}{2}\end{bmatrix}.$$

52. Haec autem formula pro Q inventa omnino similis est illi, quam pro P invenimus, hoc tantum discrimine, quod litterae p et q inter se sunt permutatae; quocirca, si integratio per seriem instituatur, proveniet sequens

$$Q = \frac{1}{\sqrt[n]{2^q}} \left\{ \begin{array}{l} \frac{1}{q} + \frac{n-p}{2n} \cdot \frac{1}{n+q} + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{1}{2n+q} \\ \quad + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+q} + \text{etc.} \end{array} \right\},$$

quae series aeque converget ac praecedens pro P inventa. His autem duabus seriebus ad calculum revocatis semper erit valor quaesitus

$$S = P + Q.$$

COROLLARIUM 1

53. Iste calculus plurimum contrahetur iis casibus, quibus est $p = q$; tum enim fiet $P = Q$ hisque casibus, quibus

$$S = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}},$$

valor istius formulae ab $x = 0$ ad $x = 1$ extensae erit

$$S = \frac{2}{\sqrt[n]{2^p}} \left\{ \begin{array}{l} \frac{1}{p} + \frac{n-p}{2n} \cdot \frac{1}{n+p} + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{1}{2n+p} \\ \quad + \frac{n-p}{2n} \cdot \frac{2n-p}{4n} \cdot \frac{3n-p}{6n} \cdot \frac{1}{3n+p} + \text{etc.} \end{array} \right\}.$$

COROLLARIUM 2

54. Quoniam igitur in singulis ordinibus nonnullae huiusmodi formulae (p, p) occurrunt, statim atque valores aliquot huiusmodi formularum fuerint ad calculum decimalem revocati, quoniam formulae circulares per se sunt notae, ex iis valores omnium reliquarum formularum eiusdem ordinis assignare licebit.

EXEMPLUM

55. Proposita sit formula ordinis primi, ubi $p = q = 2$ et

$$S = \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}}.$$

Series igitur pro S inventa erit

$$S=\sqrt[3]{2}\left(\frac{1}{2}+\frac{1}{6}\cdot\frac{1}{5}+\frac{1}{6}\cdot\frac{4}{12}\cdot\frac{1}{8}+\frac{1}{6}\cdot\frac{4}{12}\cdot\frac{7}{18}\cdot\frac{1}{11}+\frac{1}{6}\cdot\frac{4}{12}\cdot\frac{7}{18}\cdot\frac{10}{24}\cdot\frac{1}{14}+\text{etc.}\right).$$

Subducto autem calculo reperitur

$$S=0{,}54326\sqrt[3]{2}=0{,}68446,$$

qui ergo est valor formulae $(2,2)$ in ordine primo (§ 22), ubi invenimus $(2,2)=\frac{A}{P}$, ita ut iam sit $P=\frac{A}{(2,2)}$. Est vero

$$A=\frac{2\pi}{3\sqrt{3}}=1{,}20920,$$

hinc erit

$$P=1{,}76664=(1,1),$$

unde in fractionibus decimalibus ternae formulae ordinis primi erunt

$$(1,1)=1{,}76664,\quad (1,2)=1{,}20920,\quad (2,2)=0{,}68446.^{1)}$$

Hocque modo etiam omnes formulas sequentium ordinum evolvere licebit.

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1) Editio princeps: *Subducto autem calculo reperitur*

$$S=0{,}54325\sqrt[3]{2}=0{,}68445,$$

qui ergo ... *Est vero* $A=\frac{2\pi}{3\sqrt{3}}=1{,}20918$, *hinc erit* $P=2{,}22582=(1,1)$, *unde in* ... *primi erunt*

$$(1,1)=2{,}22582,\quad (1,2)=1{,}20918,\quad (2,2)=0{,}68445.$$

Correxit A. L.

ADDITAMENTUM AD DISSERTATIONEM DE VALORIBUS FORMULAE INTEGRALIS

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

AB $x=0$ AD $x=1$ EXTENSAE[1]

Conventui exhibitum die 17. Octobris 1776

Nova acta academiae scientiarum Petropolitanae 5 (1787), 1789, p. 118—129
Summarium ibidem p. 72—73

SUMMARIUM

Les Géomètres ne méconnoîtront pas dans ce supplément un trait caractéristique du génie de feu M. Euler. Ils savent qu'il est peu de sujets pour lesquels il ne soit revenu sur ses traces, en donnant toujours un plus haut degré de perfection à tout ce qu'il avoit fait antérieurement. Nos Extraits mêmes en fournissent plus d'une preuve, où nous nous attachons principalement à indiquer, souvent à la vérité par peu de traits, ce que l'illustre Auteur avoit fait autrefois dans la même matière.

M. Euler s'étoit vu arrêté dans le cours de son Mémoire précédent par les difficultés que le grand nombre d'équations fait naître, dès que l'on veut donner à l'exposant n une valeur qui surpasse 7; c'est pourquoi il n'avoit poursuivi ses recherches que jusqu'au cinquième ordre. Ayant vu cependant que de ce grand nombre d'équations, qui résultent dans chaque ordre, toutes ne sont pas nécessaires à la détermination des formules contenues dans cet ordre, il a voulu examiner le huitième, où $n=10$, en ne tenant compte que des équations qui concourent à la détermination des formules de cet ordre. De cette façon l'Auteur trouve dans cet ordre 45 formules dont neuf, savoir cinq circulaires et quatre transcendantes, servent à déterminer les 36 autres; et cette méthode peut être employée avec le même avantage pour les ordres supérieurs.

Le Mémoire est terminé par une méthode générale de traiter l'ordre n.

1) Vide Commentationem 640, p. 392. Indicis Enestroemiani numerus 640 etiam pro hoc Additamento valet. A. L.

1. Si methodum in praecedente dissertatione traditam ad altiores ordines quam $n = 7$ transferre vellemus, ob ingentem aequationum considerandarum numerum labor fieret nimis molestus. Quoniam autem vidimus non omnes istas aequationes concurrere ad valores singularum formularum determinandos, opus non mediocriter sublevabitur, si quovis casu eas tantum aequationes in computum ducamus, quae immediate ad determinationes formularum perducant, quemadmodum hic pro casu $n = 10$ sum ostensurus.

DETERMINATIO
HARUM FORMULARUM PRO CASU $n = 10$ UBI FORMULA

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[10]{(1-x^{10})^{10-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[10]{(1-x^{10})^{10-p}}}$$

2. Hoc casu ergo formulae valorem absolutum recipientes sunt

$$(10, 1) = 1, \quad (10, 2) = \frac{1}{2}, \quad (10, 3) = \frac{1}{3} \quad \text{et in genere} \quad (10, \alpha) = \frac{1}{\alpha}.$$

Deinde omnes formulae, in quibus est $p + q = 10$, a circulo pendent ideoque pro cognitis haberi possunt, quas ergo propriis litteris designemus:

$$(1, 9) = \frac{\pi}{10 \sin. \frac{1}{10}\pi} = A, \qquad (6, 4) = \frac{\pi}{10 \sin. \frac{6}{10}\pi} = D,$$

$$(2, 8) = \frac{\pi}{10 \sin. \frac{2}{10}\pi} = B, \qquad (7, 3) = \frac{\pi}{10 \sin. \frac{7}{10}\pi} = C,$$

$$(3, 7) = \frac{\pi}{10 \sin. \frac{3}{10}\pi} = C, \qquad (8, 2) = \frac{\pi}{10 \sin. \frac{8}{10}\pi} = B,$$

$$(4, 6) = \frac{\pi}{10 \sin. \frac{4}{10}\pi} = D, \qquad (9, 1) = \frac{\pi}{10 \sin. \frac{9}{10}\pi} = A.$$

$$(5, 5) = \frac{\pi}{10 \sin. \frac{5}{10}\pi} = E,$$

3. Per has autem formulas circulares reliquas in forma generali contentas neutiquam determinare licet, sed insuper aliquot formulas transcendentes in subsidium vocari oportet, ex quibus cum circularibus illis coniunctis

reliquarum omnium valores assignare licebit. Nostro autem casu, quo $n = 10$, sequentes formulas tamquam cognitas spectari conveniet, quae in ordine praecedenti, ubi $n = 9$, erant circulares, nunc autem in ordinem transcendentium transeunt. Eas igitur sequenti modo designemus

$$(1, 8) = P, \quad (2, 7) = Q, \quad (3, 6) = R, \quad (4, 5) = S,$$
$$(5, 4) = S, \quad (6, 3) = R, \quad (7, 2) = Q, \quad (8, 1) = P.$$

Scilicet si valores harum litterarum quoque tamquam cognitos spectemus, per eos cum circularibus iunctos reliquas formulas omnes in hoc ordine contentas determinare poterimus. Cum igitur numerus omnium formularum integralium in hoc ordine $n = 10$ contentarum sit 45, ex iis autem novem ut cognitae spectentur, reliquae 36 per has litteras maiusculas determinari debebunt.

4. Istas autem determinationes ex aequatione generali supra [§ 10] demonstrata peti oportet, quae hac forma continetur

$$(a, b)\,(a + b, c) = (a, c)\,(a + c, b),$$

ubi assumere licebit semper esse $b > c$, quoniam, si foret $c = b$, aequatio foret identica. Primo igitur, ut hinc aequationes, quae immediate determinationes praebeant, nanciscamur, sumamus $a + b = 10$, ut sit $(10, c) = \frac{1}{c}$; tum vero capiatur $c = b - 1$, quo facto pro a ordine scribendo numeros 1, 2, 3 etc. sequentes prodibunt determinationes

$$(1, 9)\,(10, 8) = (1, 8)\,(9, 9) \quad \text{sive} \quad \tfrac{1}{8}A = P(9, 9), \quad \text{ergo} \quad (9, 9) = \frac{A}{8P},$$
$$(2, 8)\,(10, 7) = (2, 7)\,(9, 8) \quad \text{sive} \quad \tfrac{1}{7}B = Q(9, 8), \quad \text{ergo} \quad (9, 8) = \frac{B}{7Q},$$
$$(3, 7)\,(10, 6) = (3, 6)\,(9, 7) \quad \text{sive} \quad \tfrac{1}{6}C = R(9, 7), \quad \text{ergo} \quad (9, 7) = \frac{C}{6R},$$
$$(4, 6)\,(10, 5) = (4, 5)\,(9, 6) \quad \text{sive} \quad \tfrac{1}{5}D = S(9, 6), \quad \text{ergo} \quad (9, 6) = \frac{D}{5S},$$
$$(5, 5)\,(10, 4) = (5, 4)\,(9, 5) \quad \text{sive} \quad \tfrac{1}{4}E = S(9, 5), \quad \text{ergo} \quad (9, 5) = \frac{E}{4S},$$
$$(6, 4)\,(10, 3) = (6, 3)\,(9, 4) \quad \text{sive} \quad \tfrac{1}{3}D = R(9, 4), \quad \text{ergo} \quad (9, 4) = \frac{D}{3R},$$
$$(7, 3)\,(10, 2) = (7, 2)\,(9, 3) \quad \text{sive} \quad \tfrac{1}{2}C = Q(9, 3), \quad \text{ergo} \quad (9, 3) = \frac{C}{2Q},$$
$$(8, 2)\,(10, 1) = (8, 1)\,(9, 2) \quad \text{sive} \quad B = P(9, 2), \quad \text{ergo} \quad (9, 2) = \frac{B}{P}.$$

5. Ex formulis igitur incognitis illis numero 36 iam octo determinavimus quae nobis viam sternent ad novas determinationes, quas primo derivabimus ex aequatione generali sumendo $a=1$, $b=9$ et pro c scribendo ordine numeros 1, 2, 3, ... 8, unde calculus ita se habebit:

$(1, 9)(10, 1) = (1, 1)(2, 9)$	$A = (1, 1)\frac{B}{P}$, ergo $(1, 1) = \frac{AP}{B}$,
$(1, 9)(10, 2) = (1, 2)(3, 9)$	$\frac{1}{2}A = (1, 2)\frac{C}{2Q}$, ergo $(1, 2) = \frac{AQ}{C}$,
$(1, 9)(10, 3) = (1, 3)(4, 9)$	$\frac{1}{3}A = (1, 3)\frac{D}{3R}$, ergo $(1, 3) = \frac{AR}{D}$,
$(1, 9)(10, 4) = (1, 4)(5, 9)$	$\frac{1}{4}A = (1, 4)\frac{E}{4S}$, ergo $(1, 4) = \frac{AS}{E}$,
$(1, 9)(10, 5) = (1, 5)(6, 9)$	$\frac{1}{5}A = (1, 5)\frac{D}{5S}$, ergo $(1, 5) = \frac{AS}{D}$,
$(1, 9)(10, 6) = (1, 6)(7, 9)$	$\frac{1}{6}A = (1, 6)\frac{C}{6R}$, ergo $(1, 6) = \frac{AR}{C}$,
$(1, 9)(10, 7) = (1, 7)(8, 9)$	$\frac{1}{7}A = (1, 7)\frac{B}{7Q}$, ergo $(1, 7) = \frac{AQ}{B}$,
$(1, 9)(10, 8) = (1, 8)(9, 9)$	$\frac{1}{8}A = (1, 8)\frac{A}{8P}$, ergo $(1, 8) = \frac{AP}{A}$;

hocque modo septem novas determinationes sumus adepti.

6. His autem inventis consideremus aequationes ex valoribus $a=1$ $b=8$, $c=1, 2, 3, \ldots 7$ ortas eritque

$(1, 8)(9, 1) = (1, 1)(2, 8)$	$AP = (1, 1)B$	identica,
$(1, 8)(9, 2) = (1, 2)(3, 8)$	$B = (3, 8)\frac{AQ}{C}$	$(3, 8) = \frac{BC}{AQ}$,
$(1, 8)(9, 3) = (1, 3)(4, 8)$	$\frac{CP}{2Q} = (4, 8)\frac{AR}{D}$	$(4, 8) = \frac{CDP}{2AQR}$,
$(1, 8)(9, 4) = (1, 4)(5, 8)$	$\frac{DP}{3R} = (5, 8)\frac{AS}{E}$	$(5, 8) = \frac{DEP}{3ARS}$,
$(1, 8)(9, 5) = (1, 5)(6, 8)$	$\frac{EP}{4S} = (6, 8)\frac{AS}{D}$	$(6, 8) = \frac{DEP}{4ASS}$,
$(1, 8)(9, 6) = (1, 6)(7, 8)$	$\frac{DP}{5S} = (7, 8)\frac{AR}{C}$	$(7, 8) = \frac{CDP}{5ARS}$,
$(1, 8)(9, 7) = (1, 7)(8, 8)$	$\frac{CP}{6R} = (8, 8)\frac{AQ}{B}$	$(8, 8) = \frac{BCP}{6AQR}$.

7. Novas determinationes reperiemus ponendo $a=1$, $b=7$, $c=3, 4, 5, 6$; hinc enim nanciscimur sequentes determinationes

$$\begin{array}{l|l|l}
(1, 7)(8, 3)=(1, 3)(4, 7) & D=(4, 7)\frac{AR}{D} & (4, 7)=\frac{CD}{AR}, \\
(1, 7)(8, 4)=(1, 4)(5, 7) & \frac{CDP}{2BR}=(5, 7)\frac{AS}{E} & (5, 7)=\frac{CDEP}{2ABRS}, \\
(1, 7)(8, 5)=(1, 5)(6, 7) & \frac{DEPQ}{3BRS}=(6, 7)\frac{AS}{D} & (6, 7)=\frac{DDEPQ}{3ABRSS}, \\
(1, 7)(8, 6)=(1, 6)(7, 7) & \frac{DEPQ}{4BSS}=(7, 7)\frac{AR}{C} & (7, 7)=\frac{CDEPQ}{4ABRSS}.
\end{array}$$

8. Sumamus nunc $a=1$, $b=6$, $c=4, 5$ eritque

$$\begin{array}{l|l|l}
(1, 6)(7, 4)=(1, 4)(5, 6) & D=(5, 6)\frac{AS}{E} & (5, 6)=\frac{DE}{AS}, \\
(1, 6)(7, 5)=(1, 5)(6, 6) & \frac{DEP}{2BS}=(6, 6)\frac{AS}{D} & (6, 6)=\frac{DDEP}{2ABSS}.
\end{array}$$

Hactenus igitur omnes formulas (p, q) determinavimus, in quibus $p+q>10$. Ex reliquis autem, ubi $p+q<9$, iam nacti sumus istas

$$(1, 1),\ (1, 2),\ (1, 3),\ (1, 4),\ (1, 5),\ (1, 6),\ (1, 7),$$

ita ut adhuc determinandae relinquantur istae

$$\begin{array}{l}
(2, 2),\ (2, 3),\ (2, 4),\ (2, 5),\ (2, 6), \\
(3, 3),\ (3, 4),\ (3, 5), \\
(4, 4).
\end{array}$$

9. Pro his inveniendis sumamus $a=1$ et $c=1$, pro b autem ordine capiamus numeros 2, 3 etc. atque consequemur has aequationes

$$\begin{array}{l|l|l}
(1, 2)(3, 1)=(1, 1)(2, 2) & \frac{AAQR}{CD}=(2, 2)\frac{AP}{B} & (2, 2)=\frac{ABQR}{CDP}, \\
(1, 3)(4, 1)=(1, 1)(2, 3) & \frac{AARS}{DE}=(2, 3)\frac{AP}{B} & (2, 3)=\frac{ABRS}{DEP}, \\
(1, 4)(5, 1)=(1, 1)(2, 4) & \frac{AASS}{DE}=(2, 4)\frac{AP}{B} & (2, 4)=\frac{ABSS}{DEP}, \\
(1, 5)(6, 1)=(1, 1)(2, 5) & \frac{AARS}{CD}=(2, 5)\frac{AP}{B} & (2, 5)=\frac{ABRS}{CDP}, \\
(1, 6)(7, 1)=(1, 1)(2, 6) & \frac{AAQR}{BC}=(2, 6)\frac{AP}{B} & (2, 6)=\frac{ABQR}{BCP}
\end{array}$$

sicque etiamnunc determinandae restant formulae $(3, 3)$, $(3, 4)$, $(3, 5)$ et $(4, 4)$.

10. Pro his sumatur $a=1$, $c=2$ et $b=3, 4, 5$ etc.; tum enim prodibunt hae aequationes

$(1, 3)\,(4, 2) = (1, 2)\,(3, 3)$	$\frac{AABRSS}{DDEP} = (3, 3)\frac{AQ}{C}$	$(3, 3) = \frac{ABCRSS}{DDEPQ}$,
$(1, 4)\,(5, 2) = (1, 2)\,(3, 4)$	$\frac{AABRSS}{CDEP} = (3, 4)\frac{AQ}{C}$	$(3, 4) = \frac{ABRSS}{DEPQ}$,
$(1, 5)\,(6, 2) = (1, 2)\,(3, 5)$	$\frac{AAQRS}{CDP} = (3, 5)\frac{AQ}{C}$	$(3, 5) = \frac{ARS}{DP}$.

Unica ergo formula restat determinanda, scilicet $(4, 4)$, quae ex hac aequatione

$$(1, 4)\,(5, 3) = (1, 3)\,(4, 4)$$

definietur; erit enim

$$\frac{AARSS}{DEP} = (4, 4)\frac{AR}{D} \quad \text{ideoque} \quad (4, 4) = \frac{ASS}{EPP}.$$

11. Ut nunc omnes has determinationes simul aspectui exponamus, quoniam in hoc ordine $n=10$ omnino 45 formulae integrales occurrunt, si ex iis ut cognitae spectentur novem sequentes

$$(1, 9) = A, \quad (2, 8) = B, \quad (3, 7) = C, \quad (4, 6) = D, \quad (5, 5) = E,$$
$$(1, 8) = P, \quad (2, 7) = Q, \quad (3, 6) = R, \quad (4, 5) = S,$$

reliquae triginta sex ex his sequenti modo determinabuntur:

1. $(9, 9) = \frac{A}{8P}$	8. $(9, 2) = \frac{B}{P}$,
2. $(9, 8) = \frac{B}{7Q}$	9. $(1, 1) = \frac{AP}{B}$,
3. $(9, 7) = \frac{C}{6R}$	10. $(1, 2) = \frac{AQ}{C}$,
4. $(9, 6) = \frac{D}{5S}$	11. $(1, 3) = \frac{AR}{D}$,
5. $(9, 5) = \frac{E}{4S}$	12. $(1, 4) = \frac{AS}{E}$,
6. $(9, 4) = \frac{D}{3R}$	13. $(1, 5) = \frac{AS}{D}$,
7. $(9, 3) = \frac{C}{2Q}$	14. $(1, 6) = \frac{AR}{C}$,

15. $(1, 7) = \frac{AQ}{B}$

16. $(3, 8) = \frac{BC}{AQ}$

17. $(4, 7) = \frac{CD}{AR}$

18. $(5, 6) = \frac{DE}{AS}$

19. $(2, 6) = \frac{AQR}{CP}$

20. $(3, 5) = \frac{ARS}{DP}$

21. $(4, 4) = \frac{ASS}{EP}$

22. $(4, 8) = \frac{CDP}{2AQR}$

23. $(5, 8) = \frac{DEP}{3ARS}$

24. $(6, 8) = \frac{DEP}{4ASS}$

25. $(7, 8) = \frac{CDP}{5ARS}$

26. $(8, 8) = \frac{BCP}{6AQR}$,

27. $(2, 2) = \frac{ABQR}{CDP}$,

28. $(2, 3) = \frac{ABRS}{DEP}$,

29. $(2, 4) = \frac{ABSS}{DEP}$,

30. $(2, 5) = \frac{ABRS}{CDP}$,

31. $(5, 7) = \frac{CDEP}{2ABRS}$,

32. $(6, 6) = \frac{DDEP}{2ABSS}$,

33. $(3, 4) = \frac{ABRSS}{DEPQ}$,

34. $(6, 7) = \frac{DDEPQ}{3ABRSS}$,

35. $(7, 7) = \frac{CDEPQ}{4ABRSS}$,

36. $(3, 3) = \frac{ABCRSS}{DDEPQ}$.

12. Eadem methodo, qua hic usi sumus pro casu $n = 10$, haud difficile erit ordines altiores evolvere; neque tamen hinc adhuc elucet, quanam lege omnes determinationes progrediantur, quandoquidem valores certarum formularum continuo magis evadunt complicati. Ceterum valores, quos hic invenimus, omnibus aequationibus in forma generali

$$(a, b)\,(a + b, c) = (a, c)\,(a + c, b)$$

contentis satisfacere deprehenduntur, ita ut perpetuo aequatio identica resultet neque idcirco inde ulla nova relatio inter litteras nostras maiusculas deduci queat. Tandem probe hic notasse iuvabit, quod in omnibus ordinibus praeter formulas a circulo pendentes commodissime eae formulae, quae in ordine proxime praecedente erant circulares, hic etiam tamquam cognitae accipi queant, quippe quibus determinationes omnes optimo successu perfici possunt.

METHODUS GENERALIS DETERMINANDI VALORES FORMULAE

$$(p, q)=\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\int\frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

A TERMINO $x=0$ USQUE AD $x=1$ EXTENSAE

UBI PRAETER FORMULAS CIRCULUM INVOLVENTES IN QUIBUS EST $p+q=n$ ETIAM ILLAE PRO COGNITIS ACCIPIUNTUR IN QUIBUS EST $p+q=n-1$

I. Cum aequatio generalis, unde omnes hae determinationes sunt petendae, sit

$$(a, b)\,(a+b, c)=(a, c)\,(a+c, b),$$

sumatur primo $a=n-\alpha$, $b=\alpha$ et $c=\alpha-1$ eritque aequatio

$$(n-\alpha, \alpha)\,(n, \alpha-1)=(n-\alpha, \alpha-1)\,(n-1, \alpha),$$

ubi est [§ 7]

$$(n, \alpha-1)=\frac{1}{\alpha-1}.$$

In primo autem factore ob $p=n-\alpha$ et $q=\alpha$ est $p+q=n$ ideoque datur. In tertio autem factore, ubi $p=n-\alpha$ et $q=\alpha-1$, est $p+q=n-1$ ideoque pariter datur. Hinc ergo colligimus

$$(n-1, \alpha)=\frac{1}{\alpha-1}\cdot\frac{(n-\alpha, \alpha)}{(n-\alpha, \alpha-1)},$$

ubi esse debet $\alpha>1$, ita ut pro α accipi queant omnes numeri a 2 usque ad $n-1$; at vero casu $\alpha=1$ valor formulae per se est notus.

II. In aequatione generali iam sumatur $a=\beta$, $b=n-\beta-1$ et $c=1$ eritque nostra aequatio

$$(\beta, n-\beta-1)\,(n-1, 1)=(\beta, 1)\,(\beta+1, n-\beta-1),$$

ex qua aequatione colligitur

$$(\beta, 1)=\frac{(\beta, n-\beta-1)\,(n-1, 1)}{(\beta+1, n-\beta-1)},$$

ubi esse debet $\beta < n-1$, ita ut hinc omnes formulae $(\beta, 1)$ definiantur a valore $\beta = 1$ usque ad $\beta = n-1$, quo posteriore casu formula $(n-1, 1)$ per se cognoscitur.

III. Ut hinc etiam alias formas eliciamus, sumamus $a = 1$, $b = n-2$, $c = \gamma$, ut oriatur haec aequatio

$$(1, n-2)(n-1, \gamma) = (1, \gamma)(1+\gamma, n-2),$$

ubi primus factor ac tertius dantur per II, secundus vero per I; unde quartus derivatur, scilicet

$$(1+\gamma, n-2) = \frac{(1, n-2)(n-1, \gamma)}{(1, \gamma)},$$

ubi valores ipsius $1+\gamma$ a 2 usque ad $n-2$ augeri possunt. Cum igitur per I sit

$$(n-1, \gamma) = \frac{1}{\gamma-1} \cdot \frac{(n-\gamma, \gamma)}{(n-\gamma, \gamma-1)},$$

tum vero per II sit

$$(\gamma, 1) = \frac{(\gamma, n-\gamma-1)(n-1, 1)}{(\gamma+1, n-\gamma-1)},$$

his valoribus substitutis fiet

$$(n-2, 1+\gamma) = \frac{1}{\gamma-1} \cdot \frac{(1, n-2)(n-\gamma, \gamma)(\gamma+1, n-\gamma-1)}{(n-\gamma, \gamma-1)(\gamma, n-\gamma-1)(n-1, 1)}.$$

IV. Sumamus nunc $a = 1$, $b = n-3$, $c = \delta$ prodibitque haec aequatio

$$(1, n-3)(n-2, \delta) = (1, \delta)(1+\delta, n-3),$$

unde colligitur

$$(n-3, 1+\delta) = \frac{(n-3, 1)(n-2, \delta)}{(\delta, 1)},$$

ubi ergo $1+\delta$ continet numeros $2, 3, 4, \ldots n-3$, ita ut hinc excludatur [formula] $(n-3, 1)$, quae autem per II datur. At si valores ante reperti substituantur, fiet

$$(n-3, 1+\delta) = \frac{1}{\delta-2} \cdot \frac{(n-3, 2)(n-2, 1)(n-\delta+1, \delta-1)(\delta, n-\delta)(\delta+1, n-\delta-1)}{(n-2, 2)(n-\delta+1, \delta-2)(\delta-1, n-\delta)(n-1, 1)(\delta, n-\delta-1)},$$

unde patet esse debere $\delta > 2$ eodemque modo pro praecedente formula $\gamma > 1$, ita ut hic excludantur casus $(n-3, 1)$, $(n-3, 2)$, quorum quidem prior per II datur, alter vero per se.

V. Statuamus nunc $a=1$, $b=n-4$ et $c=\varepsilon$ prodibitque haec aequatio

$$(1, n-4)(n-3, \varepsilon) = (1, \varepsilon)(1+\varepsilon, n-4),$$

unde concluditur

$$(n-4, 1+\varepsilon) = \frac{(n-4, 1)(n-3, \varepsilon)}{(1, \varepsilon)};$$

ubi si loco $(n-3, \varepsilon)$ valor ante inventus substitueretur, factor absolutus ingrederetur $\frac{1}{\varepsilon-3}$, ita ut esse debeat $\varepsilon > 3$ ideoque $1+\varepsilon > 4$, unde hic excluduntur casus $(n-4, 1)$, $(n-4, 2)$, $(n-4, 3)$, quorum quidem primus ex II, tertius autem per se datur, medius vero revera manet incognitus.

VI. Statuamus porro $a=1$, $b=n-5$, $c=\zeta$ et aequatio erit

$$(1, n-5)(n-4, \zeta) = (1, \zeta)(1+\zeta, n-5),$$

unde fit

$$(n-5, 1+\zeta) = \frac{(n-5, 1)(n-4, \zeta)}{(1, \zeta)},$$

ubi ob formulam $(n-4, \zeta)$ debet esse $\zeta > 4$ ideoque $1+\zeta > 5$, unde hinc excluduntur casus $(n-5, 1)$, $(n-5, 2)$, $(n-5, 3)$, $(n-5, 4)$, quorum quidem primus ex II constat, quartus vero per se datur, ita ut hic occurrant duo casus etiamnunc incogniti $(n-5, 2)$ et $(n-5, 3)$.

VII. Simili modo si ulterius sumamus $a=1$, $b=n-6$ et $c=\eta$, prodibit

$$(n-6, 1+\eta) = \frac{(n-6, 1)(n-5, \eta)}{(1, \eta)},$$

ubi revera occurrunt tres sequentes casus $(n-6, 2)$, $(n-6, 3)$, $(n-6, 4)$, qui adhuc manent incogniti, atque hoc modo progredi licebit, quousque necesse fuerit; unde patet numerum casuum incognitorum continuo augeri, ita ut terminorum p et q alter futurus sit vel 2 vel 3 vel 4 etc., qui igitur casus adhuc definiendi restant.

VIII. Sumamus nunc primo $a=1$, $b=\theta$, $c=1$, ut aequatio nostra fiat

$$(1, \theta)(1+\theta, 1) = (1, 1)(2, \theta),$$

unde concludimus

$$(2, \theta) = \frac{(1, \theta)(1+\theta, 1)}{(1, 1)},$$

quae formula iam omnes casus exclusos suppeditat, in quibus alter terminus erat 2.

IX. Deinde sumamus $a=2$, $b=x$ et $c=1$, ut aequatio prodeat

$$(2,\, x)\,(2+x,\, 1)=(2,\, 1)\,(3,\, x),$$

unde fit

$$(3,\, x)=\frac{(2,\, x)\,(2+x,\, 1)}{(2,\, 1)};$$

ubi cum $(2, x)$ per praecedentem numerum detur, nunc etiam ii casus innotescunt, ubi alter terminus erat 3.

X. Sumatur porro $a=3$, $b=x$, $c=1$ eritque

$$(3,\, x)\,(3+x,\, 1)=(3,\, 1)\,(4,\, x),$$

unde fit

$$(4,\, x)=\frac{(3,\, x)\,(3+x,\, 1)}{(3,\, 1)},$$

unde igitur ii casus eliciuntur, ubi alter terminus erat 4.

Eodem modo pro reliquis proceditur sicque omnes plane casus in formula proposita contenti plene sunt determinati.

QUATUOR THEOREMATA MAXIME NOTATU DIGNA IN CALCULO INTEGRALI

Conventui exhibita die 1. Iulii 1776

Commentatio 651 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 7 (1789), 1793, p. 22—41
Summarium ibidem p. 37—38

SUMMARIUM

Le sujet de ce Mémoire est encore, comme celui du Mémoire précédent[1]), une suite des recherches multipliées de l'Auteur sur les courbes algébriques, dont les arcs indéfinis s sont exprimés par une même formule intégrale, savoir

$$s = \int \partial\varphi \sin.\varphi^{n-1}.$$

Car comme les ordonnées d'une courbe quelconque, qui répondent à cet arc indéfini s, sont

$$x = \int \partial s \cos.\omega \quad \text{et} \quad y = \int \partial s \sin.\omega,$$

ω étant l'angle de courbure, tout revient à trouver pour cet angle ω une valeur telle que x et y puissent être exprimés algébriquement. Or M. EULER a trouvé que la valeur

$$\omega = (n + 2i + 1)\varphi$$

satisfait à cette condition, où n désigne un nombre quelconque, entier ou fractionnaire, positif ou négatif, et i un nombre entier positif quelconque; propriété qui est démontrée dans le premier et le second Théorème, où l'Auteur donne pour $\int \partial s \sin.\omega$ et pour $\int \partial s \cos.\omega$ les intégrales algébriques composées la première des sinus, l'autre des cosinus d'angles qui

1) Memoire 650 (suivant l'Index d'ENESTRÖM): *De formulis differentialibus, quae per duas pluresve quantitates datas multiplicatae fiant integrabiles*, Nova acta acad. sc. Petrop. 7 (1789), 1793, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 23. A. L.

forment une progression arithmétique décroissante dont le premier terme est $(n+2i)\varphi$, le dernier $n\varphi$ et la différence 2φ. La démonstration de ces deux premiers Théorèmes est fondée sur une réduction générale tirée de la différentielle des deux formules

$$\sin.\varphi^n \sin.\lambda\varphi \quad \text{et} \quad \sin.\varphi^n \cos.\lambda\varphi.$$

C'est par des expressions semblables et démontrées de la même manière que dans les deux derniers Théorèmes M. EULER présente les intégrales des mêmes formules $\int \partial s \cos.\omega$ et $\int \partial s \sin.\omega$, ∂s étant $= \partial\varphi \cos.\varphi^{n-1}$, de sorte que ces deux Théorèmes combinés peuvent servir à trouver une infinité de courbes algébriques dont les arcs indéfinis sont exprimés par la même formule intégrale $\int \partial\varphi \cos.\varphi^{n-1}$, les deux premiers Théorèmes ayant fourni une infinité de courbes algébriques, dont les arcs indéfinis sont $= \int \partial\varphi \sin.\varphi^{n-1}$.

Finalement tous les quatre Théorèmes combinés frayent le chemin à la solution du Problème de trouver une infinité de courbes algébriques dont les arcs indéfinis sont exprimés plus généralement par la forme

$$\int \partial\varphi \sqrt{(aa \sin.\varphi^{2n-2} + bb \cos.\varphi^{2n-2})},$$

Problème dont la solution termine ce Mémoire.

Nous ne pouvons pas passer sous silence une chose digne d'être relevée, c'est que M. EULER a résolu au § 20, pour ainsi dire en passant, un problème qui l'avoit beaucoup occupé autrefois et dont il avoit désespéré plus d'une fois de trouver la solution.[1])

THEOREMA 1

1. *Denotante φ angulum quemcunque variabilem si n significet numerum quemcunque sive integrum sive fractum sive positivum sive negativum, tum vero statuatur*

$$\partial s = \partial\varphi \sin.\varphi^{n-1},$$

sequentes formulae integrales omnes algebraice exhiberi possunt:

$$\text{I.} \quad \int \partial s \sin.(n+1)\varphi = \frac{\sin.\varphi^n}{n} \sin.n\varphi,$$

$$\text{II.} \quad \int \partial s \sin.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\left(\sin.(n+2)\varphi + \frac{1}{n}\sin.n\varphi\right),$$

1) Voir le mémoire 639 (suivant l'Index d'ENESTRÖM): *De innumeris curvis algebraicis, quarum longitudinem per arcus ellipticos metiri licet*, Nova acta acad. sc. Petrop. **5** (1787), 1789, p. 71; *LEONHARDI EULERI Opera omnia*, series I, vol. 21, p. 163, surtout p. 178. A. L.

$$\text{III.}\quad \int \partial s \sin.(n+5)\varphi$$

$$= \frac{\sin.\varphi^n}{n+2}\left(\sin.(n+4)\varphi + \frac{2}{n+1}\sin.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \sin.(n+7)\varphi$$

$$= \frac{\sin.\varphi^n}{n+3}\left(\sin.(n+6)\varphi + \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \sin.(n+9)\varphi$$

$$= \frac{\sin.\varphi^n}{n+4}\left(\sin.(n+8)\varphi + \frac{4}{n+3}\sin.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi\right.$$
$$\left. + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{VI.}\quad \int \partial s \sin.(n+11)\varphi$$

$$= \frac{\sin.\varphi^n}{n+5}\left(\sin.(n+10)\varphi + \frac{5}{n+4}\sin.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\sin.(n+6)\varphi\right.$$
$$+ \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi$$
$$\left. + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right)$$

etc.

Unde si i denotet numerum positivum quemcunque, generaliter habebimus

$$\int \partial s \sin.(n+2i+1)\varphi$$

$$= \frac{\sin.\varphi^n}{n+i}\left(\sin.(n+2i)\varphi + \frac{i}{n+i-1}\sin.(n+2i-2)\varphi\right.$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\sin.(n+2i-8)\varphi$$
$$\left. + \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cdot\frac{i-4}{n+i-5}\sin.(n+2i-10)\varphi + \text{etc.}\right),$$

quae terminorum progressio quovis casu sponte abrumpitur.

DEMONSTRATIO

2. Ad veritatem huius theorematis demonstrandam consideretur ista formula

$$Z = \sin.\varphi^n \sin.\lambda\varphi,$$

quae differentiata dat

$$\partial Z = \partial\varphi \sin.\varphi^{n-1}(n\cos.\varphi\sin.\lambda\varphi + \lambda\sin.\varphi\cos.\lambda\varphi).$$

At per reductiones cognitas est

$$\cos.\varphi\sin.\lambda\varphi = +\frac{1}{2}\sin.(\lambda-1)\varphi + \frac{1}{2}\sin.(\lambda+1)\varphi$$

et

$$\sin.\varphi\cos.\lambda\varphi = -\frac{1}{2}\sin.(\lambda-1)\varphi + \frac{1}{2}\sin.(\lambda+1)\varphi,$$

quibus valoribus substitutis, quoniam posuimus $\partial\varphi\sin.\varphi^{n-1} = \partial s$, erit

$$2\partial Z = \partial s((n-\lambda)\sin.(\lambda-1)\varphi + (n+\lambda)\sin.(\lambda+1)\varphi),$$

unde denuo per partes integrando deducimus

$$\int\partial s\sin.(\lambda+1)\varphi = \frac{2Z}{\lambda+n} + \frac{\lambda-n}{\lambda+n}\int\partial s\sin.(\lambda-1)\varphi$$

sive

$$\int\partial s\sin.(\lambda+1)\varphi = \frac{2\sin.\varphi^n\sin.\lambda\varphi}{\lambda+n} + \frac{\lambda-n}{\lambda+n}\int\partial s\sin.(\lambda-1)\varphi.$$

3. Stabilita igitur hac postrema reductione generali capiamus $\lambda = n$, ut adipiscamur istam integrationem absolutam

$$\int\partial s\sin.(n+1)\varphi = \frac{\sin.\varphi^n}{n}\sin.n\varphi.$$

Nunc vero statuamus $\lambda = n+2$ et forma illa generalis dabit

$$\int\partial s\sin.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\sin.(n+2)\varphi + \frac{1}{n+1}\int\partial s\sin.(n+1)\varphi$$

sicque haec integratio ad praecedentem est reducta. Iam ponamus $\lambda = n + 4$ et forma generalis suppeditabit

$$\int \partial s \sin. (n+5)\varphi = \frac{\sin. \varphi^n}{n+2} \sin. (n+4)\varphi + \frac{2}{n+2}\int \partial s \sin. (n+3)\varphi,$$

quae ergo integratio iterum ad praecedentem est reducta. Sit porro $\lambda = n + 6$ et ex forma generali prodibit

$$\int \partial s \sin. (n+7)\varphi = \frac{\sin. \varphi^n}{n+3} \sin. (n+6)\varphi + \frac{3}{n+3}\int \partial s \sin. (n+5)\varphi$$

sicque augendis continuo valoribus ipsius λ binario ulterius progredi licebit.

4. Quodsi iam singulos valores integrales antecedentes in sequentibus substituamus, sequentes orientur integrationes absolutae:

$$\text{I.}\quad \int \partial s \sin. (n+1)\varphi = \frac{\sin. \varphi^n}{n} \sin. n\varphi,$$

$$\text{II.}\quad \int \partial s \sin. (n+3)\varphi = \frac{\sin. \varphi^n}{n+1}\left(\sin. (n+2)\varphi + \frac{1}{n}\sin. n\varphi\right),$$

$$\text{III.}\quad \int \partial s \sin. (n+5)\varphi$$

$$= \frac{\sin. \varphi^n}{n+2}\left(\sin. (n+4)\varphi + \frac{2}{n+1}\sin. (n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin. n\varphi\right).$$

$$\text{IV.}\quad \int \partial s \sin. (n+7)\varphi$$

$$= \frac{\sin. \varphi^n}{n+3}\left(\sin.(n+6)\varphi + \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin. n\varphi\right);$$

quae cum sint eae ipsae formulae, quas in theoremate annunciavimus, eius veritas sufficienter est evicta.

THEOREMA 2

5. *Denotante φ angulum quemcunque variabilem si n denotet numerum quemcunque ac brevitatis gratia ponatur ut ante*

$$\partial s = \partial\varphi \sin.\varphi^{n-1},$$

etiam omnes sequentes integrationes per algebraicos valores exhiberi possunt:

$$\text{I.}\quad \int \partial s \cos.(n+1)\varphi = \frac{\sin.\varphi^n}{n}\cos.n\varphi,$$

$$\text{II.}\quad \int \partial s \cos.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\left(\cos.(n+2)\varphi + \frac{1}{n}\cos.n\varphi\right),$$

$$\text{III.}\quad \int \partial s \cos.(n+5)\varphi$$

$$= \frac{\sin.\varphi^n}{n+2}\left(\cos.(n+4)\varphi + \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \cos.(n+7)\varphi$$

$$= \frac{\sin.\varphi^n}{n+3}\left(\cos.(n+6)\varphi + \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \cos.(n+9)\varphi$$

$$= \frac{\sin.\varphi^n}{n+4}\left(\cos.(n+8)\varphi + \frac{4}{n+3}\cos.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi\right.$$

$$\left. + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{VI.}\quad \int \partial s \cos.(n+11)\varphi$$

$$= \frac{\sin.\varphi^n}{n+5}\left(\cos.(n+10)\varphi + \frac{5}{n+4}\cos.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cos.(n+6)\varphi\right.$$

$$+ \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi$$

$$\left. + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right)$$

etc.

Unde patet, si i denotet numerum positivum quemcunque, fore in genere

$$\int \partial s \cos.(n+2i+1)\varphi$$
$$=\frac{\sin.\varphi^n}{n+i}\Big(\cos.(n+2i)\varphi+\frac{i}{n+i-1}\cos.(n+2i-2)\varphi$$
$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi$$
$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi+\text{etc.}\Big),$$

quos terminos quovis casu eousque continuari oportet, donec sponte evanescant.

DEMONSTRATIO

6. Ad veritatem horum integralium demonstrandam consideretur ista formula

$$Z=\sin.\varphi^n\cos.\lambda\varphi,$$

cuius differentiatio praebet

$$\partial Z=\partial\varphi\sin.\varphi^{n-1}(n\cos.\varphi\cos.\lambda\varphi-\lambda\sin.\varphi\sin.\lambda\varphi),$$

quae expressio ob

$$\cos.\varphi\cos.\lambda\varphi=\frac{1}{2}\cos.(\lambda-1)\varphi+\frac{1}{2}\cos.(\lambda+1)\varphi$$

et

$$\sin.\varphi\sin.\lambda\varphi=\frac{1}{2}\cos.(\lambda-1)\varphi-\frac{1}{2}\cos.(\lambda+1)\varphi,$$

si loco $\partial\varphi\sin.\varphi^{n-1}$ valorem assumtum ∂s scribamus, fiet

$$2\partial Z=\partial s((n-\lambda)\cos.(\lambda-1)\varphi+(n+\lambda)\cos.(\lambda+1)\varphi),$$

unde iterum per partes integrando erit

$$2\sin.\varphi^n\cos.\lambda\varphi=(n-\lambda)\int\partial s\cos.(\lambda-1)\varphi+(n+\lambda)\int\partial s\cos.(\lambda+1)\varphi,$$

atque hinc deducimus sequentem reductionem generalem

$$\int\partial s\cos.(\lambda+1)\varphi=\frac{2\sin.\varphi^n\cos.\lambda\varphi}{\lambda+n}+\frac{\lambda-n}{\lambda+n}\int\partial s\cos.(\lambda-1)\varphi.$$

7. Ponamus igitur primo $\lambda = n$, ut obtineamus hanc integrationem absolutam

$$\int \partial s \cos.(n+1)\varphi = \frac{\sin.\varphi^n}{n} \cos.n\varphi.$$

Fiat iam $\lambda = n+2$ et forma generalis dabit

$$\int \partial s \cos.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1} \cos.(n+2)\varphi + \frac{1}{n+1}\int \partial s \cos.(n+1)\varphi.$$

Statuatur porro $\lambda = n+4$ et consequemur

$$\int \partial s \cos.(n+5)\varphi = \frac{\sin.\varphi^n}{n+2} \cos.(n+4)\varphi + \frac{2}{n+2}\int \partial s \cos.(n+3)\varphi.$$

Ponamus ulterius $\lambda = n+6$ ac reperiemus

$$\int \partial s \cos.(n+7)\varphi = \frac{\sin.\varphi^n}{n+3} \cos.(n+6)\varphi + \frac{3}{n+3}\int \partial s \cos.(n+5)\varphi.$$

Faciamus simili modo ulterius $\lambda = n+8$ ac nanciscemur

$$\int \partial s \cos.(n+9)\varphi = \frac{\sin.\varphi^n}{n+4} \cos.(n+8)\varphi + \frac{4}{n+4}\int \partial s \cos.(n+7)\varphi$$

etc.

8. Quodsi iam singulos valores integrales praecedentes in sequentes introducamus, perveniemus ad istas integrationes absolutas:

$$\text{I. } \int \partial s \cos.(n+1)\varphi = \frac{\sin.\varphi^n}{n} \cos.n\varphi,$$

$$\text{II. } \int \partial s \cos.(n+3)\varphi = \frac{\sin.\varphi^n}{n+1}\left(\cos.(n+2)\varphi + \frac{1}{n}\cos.n\varphi\right),$$

$$\text{III. } \int \partial s \cos.(n+5)\varphi$$
$$= \frac{\sin.\varphi^n}{n+2}\left(\cos.(n+4)\varphi + \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{IV. } \int \partial s \cos.(n+7)\varphi$$
$$= \frac{\sin.\varphi^n}{n+3}\left(\cos.(n+6)\varphi + \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right)$$

etc.,

quae manifesto sunt eae ipsae formulae, quas in theoremate produximus, quarum ergo veritas nunc solide est demonstrata.

COROLLARIUM

9. Haec duo theoremata combinata inservire possunt ad innumerabiles curvas algebraicas inveniendas, quarum arcus indefiniti s omnes per eandem formulam integralem $\int \partial\varphi \sin.\varphi^{n-1}$ exprimantur. Cum enim elementum curvae sit

$$\partial s = \partial\varphi \sin.\varphi^{n-1},$$

omnes plane curvae huic conditioni satisfacientes ita generaliter exhiberi possunt, ut earum coordinatae sint

$$x = \int \partial s \cos.\omega \quad \text{et} \quad y = \int \partial s \sin.\omega.$$

Nunc autem videmus ambas istas expressiones revera fore algebraicas, si angulus ω ita accipiatur, ut sit

$$\omega = (n + 2i + 1)\varphi,$$

ubi loco i numerum quemcunque integrum positivum accipere licet. Quamobrem numerum talium curvarum algebraicarum in infinitum augere licebit; curva autem simplicissima sine dubio prodibit ponendo $i = 0$. Hoc argumentum iam nuper fusius pertractavimus.[1])

THEOREMA 3

10. *Denotante φ angulum quemcunque variabilem si n significet numerum quemcunque sive integrum sive fractum sive positivum sive negativum, tum vero statuatur*

$$\partial s = \partial\varphi \cos.\varphi^{n-1},$$

sequentes formulae integrales omnes algebraice exhiberi possunt:

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1) Confer Commentationem 645 (indicis Eneströmiani): *De curvis algebraicis, quarum longitudo exprimitur hac formula integrali* $\int \frac{v^{m-1}\partial v}{\sqrt{(1-v^{2n})}}$, Nova acta acad. sc. Petrop. 6 (1788), 1790, p. 36; *Leonhardi Euleri Opera omnia*, series I, vol. 21, p. 180. A. L.

$$\text{I.}\quad \int \partial s \cos.(n+1)\varphi = \frac{\cos.\varphi^n}{n} \sin.n\varphi,$$

$$\text{II.}\quad \int \partial s \cos.(n+3)\varphi = \frac{\cos.\varphi^n}{n+1}\left(\sin.(n+2)\varphi - \frac{1}{n}\sin.n\varphi\right),$$

$$\text{III.}\quad \int \partial s \cos.(n+5)\varphi$$

$$= \frac{\cos.\varphi^n}{n+2}\left(\sin(n+4)\varphi - \frac{2}{n+1}\sin.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \cos.(n+7)\varphi$$

$$= \frac{\cos.\varphi^n}{n+3}\left(\sin.(n+6)\varphi - \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \cos.(n+9)\varphi$$

$$= \frac{\cos.\varphi^n}{n+4}\left(\sin.(n+8)\varphi - \frac{4}{n+3}\sin.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi\right.$$

$$\left. - \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{VI.}\quad \int \partial s \cos.(n+11)\varphi$$

$$= \frac{\cos.\varphi^n}{n+5}\left(\sin.(n+10)\varphi - \frac{5}{n+4}\sin.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\sin.(n+6)\varphi\right.$$

$$- \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi$$

$$\left. - \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right).$$

Ex quibus concluditur fore generaliter denotante i numerum integrum positivum quemcunque

$$\int \partial s \cos.(n+2i+1)\varphi$$

$$= \frac{\cos.\varphi^n}{n+i}\left(\sin.(n+2i)\varphi - \frac{i}{n+i-1}\sin.(n+2i-2)\varphi\right.$$

$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi$$

$$\left. - \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi + \text{etc.}\right).$$

DEMONSTRATIO

11. Ad veritatem huius theorematis demonstrandam consideretur ista formula

$$Z = \cos. \varphi^n \sin. \lambda\varphi,$$

quae differentiata dat

$$\partial Z = \partial\varphi \cos. \varphi^{n-1}(-n \sin. \varphi \sin. \lambda\varphi + \lambda \cos. \varphi \cos. \lambda\varphi),$$

quae per reductiones ante adhibitas transformatur in hanc formam

$$2\partial Z = \partial s((\lambda - n) \cos. (\lambda - 1)\varphi + (\lambda + n) \cos. (\lambda + 1)\varphi),$$

unde iterum per partes integrando nanciscimur

$$2Z = (\lambda - n)\int \partial s \cos. (\lambda - 1)\varphi + (\lambda + n)\int \partial s \cos. (\lambda + 1)\varphi,$$

hincque deducimus istam integrationem generalem

$$\int \partial s \cos. (\lambda + 1)\varphi = \frac{2 \cos. \varphi^n \sin. \lambda\varphi}{\lambda + n} - \frac{\lambda - n}{\lambda + n}\int \partial s \cos. (\lambda - 1)\varphi.$$

12. Sumamus nunc primo $\lambda = n$, ut posterius integrale tollatur, ac prodibit

$$\int \partial s \cos. (n + 1)\varphi = \frac{\cos. \varphi^n}{n} \sin. n\varphi.$$

Nunc autem porro ponamus $\lambda = n + 2$ et forma nostra generalis nobis praebebit

$$\int \partial s \cos. (n + 3)\varphi = \frac{\cos. \varphi^n}{n + 1} \sin. (n + 2)\varphi - \frac{1}{n + 1}\int \partial s \cos. (n + 1)\varphi,$$

ubi ergo posterius integrale iam est inventum. Fiat ulterius $\lambda = n + 4$ et habebimus

$$\int \partial s \cos. (n + 5)\varphi = \frac{\cos. \varphi^n}{n + 2} \sin. (n + 4)\varphi - \frac{2}{n + 2}\int \partial s \cos. (n + 3)\varphi,$$

quod postremum integrale itidem iam patet. Sumamus nunc $\lambda = n + 6$ et forma generalis dabit

$$\int \partial s \cos. (n + 7)\varphi = \frac{\cos. \varphi^n}{n + 3} \sin. (n + 6)\varphi - \frac{3}{n + 3}\int \partial s \cos. (n + 5)\varphi.$$

Simili modo si faciamus $\lambda = n + 8$, obtinebimus

$$\int \partial s \cos.(n+9)\varphi = \frac{\cos.\varphi^n}{n+4}\sin.(n+8)\varphi - \frac{4}{n+4}\int \partial s \cos.(n+7)\varphi.$$

Hocque modo ulterius progrediendo perpetuo sequentia integralia per praecedentia exprimere licebit.

13. Quodsi ergo valores integrales praecedentes in sequentibus substituamus, consequemur istas integrationes absolutas:

$$\text{I.}\quad \int \partial s \cos.(n+1)\varphi = \frac{\cos.\varphi^n}{n}\sin.n\varphi,$$

$$\text{II.}\quad \int \partial s \cos.(n+3)\varphi = \frac{\cos.\varphi^n}{n+1}\left(\sin.(n+2)\varphi - \frac{1}{n}\sin.n\varphi\right),$$

$$\text{III.}\quad \int \partial s \cos.(n+5)\varphi$$

$$= \frac{\cos.\varphi^n}{n+2}\left(\sin.(n+4)\varphi - \frac{2}{n+1}\sin.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \cos.(n+7)\varphi$$

$$= \frac{\cos.\varphi^n}{n+3}\left(\sin.(n+6)\varphi - \frac{3}{n+2}\sin.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \cos.(n+9)\varphi$$

$$= \frac{\cos.\varphi^n}{n+4}\left(\sin.(n+8)\varphi - \frac{4}{n+3}\sin.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\sin.(n+4)\varphi \right.$$

$$\left. - \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\sin.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\sin.n\varphi\right).$$

etc.,

unde veritas nostri theorematis abunde elucet.

THEOREMA 4

14. *Denotante φ angulum quemcunque variabilem si n significet numerum quemcunque sive integrum sive fractum sive positivum sive negativum, tum vero statuatur*

$$\partial s = \partial \varphi \cos. \varphi^{n-1},$$

sequentes formulae integrales omnes algebraice exprimi poterunt:

$$\text{I.}\quad \int \partial s \sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos. n\varphi,$$

$$\text{II.}\quad \int \partial s \sin.(n+3)\varphi = -\frac{\cos.\varphi^n}{n+1}\left(\cos.(n+2)\varphi - \frac{1}{n}\cos. n\varphi\right),$$

$$\text{III.}\quad \int \partial s \sin.(n+5)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+2}\left(\cos.(n+4)\varphi - \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos. n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \sin.(n+7)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+3}\left(\cos.(n+6)\varphi - \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos. n\varphi\right),$$

$$\text{V.}\quad \int \partial s \sin.(n+9)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+4}\left(\cos.(n+8)\varphi - \frac{4}{n+3}\cos.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi\right.$$

$$\left. - \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos. n\varphi\right),$$

$$\text{VI.}\quad \int \partial s \sin.(n+11)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+5}\left(\cos.(n+10)\varphi - \frac{5}{n+4}\cos.(n+8)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cos.(n+6)\varphi\right.$$

$$- \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi + \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi$$

$$\left. - \frac{5}{n+4}\cdot\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos. n\varphi\right).$$

Unde manifesto patet, si i denotet numerum quemcunque integrum positivum, fore in genere

$$\int \partial s \sin.(n+2i+1)\varphi$$
$$= -\frac{\cos.\varphi^n}{n+i}\Big(\cos.(n+2i)\varphi - \frac{i}{n+i-1}\cos.(n+2i-2)\varphi$$
$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi$$
$$-\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi$$
$$+\frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cos.(n+2i-8)\varphi - \text{etc.}\Big).$$

DEMONSTRATIO

15. Ad hoc theorema demonstrandum consideretur formula

$$Z = \cos.\varphi^n \cos.\lambda\varphi,$$

quae differentiata praebet

$$\partial Z = -\partial\varphi \cos.\varphi^{n-1}(n\sin.\varphi\cos.\lambda\varphi + \lambda\cos.\varphi\sin.\lambda\varphi),$$

quae per notas reductiones reducitur ad hanc formam

$$2\partial Z = -\partial s((\lambda+n)\sin.(\lambda+1)\varphi + (\lambda-n)\sin.(\lambda-1)\varphi),$$

quae iterum per partes integrata dat

$$2Z = -(\lambda+n)\int\partial s\sin.(\lambda+1)\varphi - (\lambda-n)\int\partial s\sin.(\lambda-1)\varphi,$$

unde deducitur ista integratio generalis

$$\int\partial s\sin.(\lambda+1)\varphi = -\frac{2\cos.\varphi^n\cos.\lambda\varphi}{\lambda+n} - \frac{\lambda-n}{\lambda+n}\int\partial s\sin.(\lambda-1)\varphi.$$

16. Ut membrum integrale postremum e medio tollatur, capiamus $\lambda = n$ et forma generalis dabit

$$\int\partial s\sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos.n\varphi.$$

Statuamus nunc porro $\lambda = n + 2$ ac proveniet

$$\int \partial s \sin.(n+3)\varphi = -\frac{\cos.\varphi^n}{n+1}\cos.(n+2)\varphi - \frac{1}{n+1}\int \partial s \sin.(n+1)\varphi.$$

Fiat porro $\lambda = n + 4$, ut oriatur

$$\int \partial s \sin.(n+5)\varphi = -\frac{\cos.\varphi^n}{n+2}\cos.(n+4)\varphi - \frac{2}{n+2}\int \partial s \sin.(n+3)\varphi.$$

Sit iam $\lambda = n + 6$; fiet

$$\int \partial s \sin.(n+7)\varphi = -\frac{\cos.\varphi^n}{n+3}\cos.(n+6)\varphi - \frac{3}{n+3}\int \partial s \sin.(n+5)\varphi.$$

Simili modo sit $\lambda = n + 8$ ac resultabit

$$\int \partial s \sin.(n+9)\varphi = -\frac{\cos.\varphi^n}{n+4}\cos.(n+8)\varphi - \frac{4}{n+4}\int \partial s \sin.(n+7)\varphi$$

etc.,

ubi pariter sequentia integralia per praecedentia definiuntur.

17. Quamobrem si ubique valores integrales praecedentes substituantur, orientur sequentes integrationes absolutae:

$$\text{I.}\quad \int \partial s \sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos.n\varphi,$$

$$\text{II.}\quad \int \partial s \sin.(n+3)\varphi = -\frac{\cos.\varphi^n}{n+1}\left(\cos.(n+2)\varphi - \frac{1}{n}\cos.n\varphi\right),$$

$$\text{III.}\quad \int \partial s \sin.(n+5)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+2}\left(\cos.(n+4)\varphi - \frac{2}{n+1}\cos.(n+2)\varphi + \frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{IV.}\quad \int \partial s \sin.(n+7)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+3}\left(\cos.(n+6)\varphi - \frac{3}{n+2}\cos.(n+4)\varphi + \frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi - \frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\right),$$

$$\text{V.}\quad \int \partial s \sin.(n+9)\varphi$$

$$= -\frac{\cos.\varphi^n}{n+4}\Big(\cos.(n+8)\varphi - \frac{4}{n+3}\cos.(n+6)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cos.(n+4)\varphi$$

$$-\frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cos.(n+2)\varphi + \frac{4}{n+3}\cdot\frac{3}{n+2}\cdot\frac{2}{n+1}\cdot\frac{1}{n}\cos.n\varphi\Big)$$

etc.

sicque veritas theorematis propositi sufficienter est evicta.

COROLLARIUM 1

18. Si $\partial s = \partial\varphi \cos.\varphi^{n-1}$ denotet elementum cuiuspiam lineae curvae, cuius coordinatae orthogonales sint x et y, ita ut sit

$$\partial s^2 = \partial x^2 + \partial y^2,$$

huic conditioni generatim satisfiet sumendo

$$\partial x = \partial s \cos.\omega \quad \text{et} \quad \partial y = \partial s \sin.\omega.$$

Nunc igitur ex binis posterioribus theorematibus patet innumerabiles huiusmodi curvas algebraicas exhiberi posse, si scilicet capiatur

$$\omega = (n + 2i + 1)\varphi,$$

quandoquidem hinc valores ipsarum x et y algebraice exprimi possunt; ac simplicissima quidem curva prodibit ponendo $i = 0$; tum enim fiet

$$x = \int \partial s \cos.(n+1)\varphi = +\frac{\cos.\varphi^n}{n}\sin.n\varphi$$

et

$$y = \int \partial s \sin.(n+1)\varphi = -\frac{\cos.\varphi^n}{n}\cos.n\varphi.$$

COROLLARIUM 2

19. Quodsi sumatur $n = 1$, ut fieri debeat $\partial s = \partial\varphi$ ideoque $s = \varphi$, hoc est arcui circulari aequalis, tum facile ostendi potest, quicunque valor numero i tribuatur, curvas resultantes omnes fore circulos, ita ut hoc casu praeter cir-

culum nulla alia curva algebraica satisfaciat, id quod pro casu $i = 3$ ostendisse sufficiat. Tum enim erit

$$x = \int \partial s \cos. 8\varphi = \frac{1}{4} \cos. \varphi (\sin. 7\varphi - \sin. 5\varphi + \sin. 3\varphi - \sin. \varphi),$$

quae forma per reductiones abit in hanc

$$x = \frac{1}{8} \sin. 8\varphi.$$

Tum vero habebitur simili modo

$$y = \int \partial s \sin. 8\varphi = -\frac{1}{4} \cos. \varphi (\cos. 7\varphi - \cos. 5\varphi + \cos. 3\varphi - \cos. \varphi),$$

quae per similes reductiones praebet

$$y = \frac{1}{8}(1 - \cos. 8\varphi) \quad \text{ideoque} \quad \frac{1}{8} - y = \frac{1}{8} \cos. 8\varphi.$$

Ex his iam valoribus coniunctis manifestum est fore

$$xx + \left(\frac{1}{8} - y\right)^2 = \frac{1}{64},$$

quae utique est aequatio pro circulo. Eodem modo ostendi potest, quicunque valor numero i tribuatur, semper quoque circulum esse proditurum.

COROLLARIUM 3

20. Casus quoque, quo $n = -\frac{1}{2}$, omni attentione est dignus, pro quo curva simplicissima erit

$$x = \int \partial s \cos. \frac{1}{2}\varphi = \frac{2 \sin. \frac{1}{2}\varphi}{\sqrt{\cos. \varphi}} \quad \text{et} \quad y = \int \partial s \sin. \frac{1}{2}\varphi = \frac{2 \cos. \frac{1}{2}\varphi}{\sqrt{\cos. \varphi}},$$

ita ut elementum huius curvae futurum sit

$$\partial s = \frac{\partial \varphi}{\cos. \varphi \sqrt{\cos. \varphi}}.$$

Iam ad angulum φ eliminandum quoniam est

$$\left(\cos. \tfrac{1}{2}\varphi\right)^2 - \left(\sin. \tfrac{1}{2}\varphi\right)^2 = \cos. \varphi,$$

habebimus

$$yy - xx = 4 \quad \text{sive} \quad yy = 4 + xx,$$

quae est aequatio pro hyperbola aequilatera sive rectangula.[1])

SCHOLION 1

21. Quamquam autem in his quatuor theorematibus infinitae formulae integrabiles sunt exhibitae, tamen occurrere possunt certi casus, quibus integralia assignata evadunt incongrua atque adeo naturam quantitatum algebraicarum penitus amittunt. Tales casus oriuntur, quoties exponens n vel evanescit vel numero integro negativo fit aequalis. Hoc enim casu fieri potest, ut quispiam factor in denominatoribus in nihilum abeat, ideoque ipsi termini in infinitum excrescere videntur. Etiamsi enim hoc incommodum adiectione constantium pariter infinitarum evitari posset, tamen ipsi termini inde resultantes non amplius forent algebraici. Ita si esset $n = 0$, omnia prorsus integralia ibi exhibita penitus tollerentur. Si autem esset $n = -1$, tum tantum primae formulae relinquerentur, sequentes omnes autem evaderent inutiles. Si esset $n = -2$, tum binae priores formae tantum subsistere possent, solae autem ternae, si esset $n = -3$, etc. His autem casibus exceptis quicunque valores exponenti n tribuantur, singula theoremata innumerabiles suppeditant formulas integrabiles.

SCHOLION 2

22. Quemadmodum binis prioribus theorematibus iam sum usus ad innumerabiles curvas algebraicas inveniendas, quarum longitudo s hoc valore exprimatur

$$s = \int \partial\varphi \sin. \varphi^{n-1},$$

ita etiam bina posteriora theoremata innumerabilibus curvis algebraicis inveniendis inservire possunt, quarum longitudo sit

$$s = \int \partial\varphi \cos. \varphi^{n-1}.$$

1) Vide notam p. 436. A. L.

Etiamsi enim hi duo casus prorsus inter se conveniant, si quidem loco φ scribendo $90^0 - \varphi$ altera formula in alteram transformatur, unde quis suspicari posset duo posteriora theoremata tuto omitti potuisse, tamen hos casus non tam plane ex prioribus deducere licet, quippe qui veritates per se notatu dignissimas involvere sunt censendi. Quin etiam omnia haec quatuor theoremata iunctim sumpta viam sternunt ad infinitas curvas algebraicas investigandas, quarum longitudo s formula multo magis complicata exprimatur; ad quod ostendendum ante oculos exponamus integrationes generales, ad quas singula theoremata nos duxerunt.

$$\text{I.}\quad \int \partial\varphi \sin.\varphi^{n-1} \sin.(n+2i+1)\varphi$$
$$\begin{aligned}= \frac{\sin.\varphi^n}{n+i}\Big(&\sin.(n+2i)\varphi + \frac{i}{n+i-1}\sin.(n+2i-2)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\sin.(n+2i-8)\varphi + \text{etc.}\Big).\end{aligned}$$

$$\text{II.}\quad \int \partial\varphi \sin.\varphi^{n-1} \cos.(n+2i+1)\varphi$$
$$\begin{aligned}= \frac{\sin.\varphi^n}{n+i}\Big(&\cos.(n+2i)\varphi + \frac{i}{n+i-1}\cos.(n+2i-2)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cos.(n+2i-8)\varphi + \text{etc.}\Big).\end{aligned}$$

$$\text{III.}\quad \int \partial\varphi \cos.\varphi^{n-1} \cos.(n+2i+1)\varphi$$
$$\begin{aligned}= \frac{\cos.\varphi^n}{n+i}\Big(&\sin.(n+2i)\varphi - \frac{i}{n+i-1}\sin.(n+2i-2)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\sin.(n+2i-4)\varphi \\ &- \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\sin.(n+2i-6)\varphi \\ &+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\sin.(n+2i-8)\varphi - \text{etc.}\Big).\end{aligned}$$

$$\text{IV. } \int \partial\varphi \cos.\varphi^{n-1} \sin.(n+2i+1)\varphi$$
$$= -\frac{\cos.\varphi^n}{n+i}\Big(\cos.(n+2i)\varphi - \frac{i}{n+i-1}\cos.(n+2i-2)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cos.(n+2i-4)\varphi$$
$$- \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cos.(n+2i-6)\varphi$$
$$+ \frac{i}{n+i-1}\cdot\frac{i-1}{n+i-2}\cdot\frac{i-2}{n+i-3}\cdot\frac{i-3}{n+i-4}\cos.(n+2i-8)\varphi - \text{etc.}\Big).$$

PROBLEMA SINGULARE

23. *Invenire innumerabiles curvas algebraicas, quarum arcus indefiniti s ista formula integrali exprimantur*

$$s = \int \partial\varphi \sqrt{(aa \sin.\varphi^{2n-2} + bb \cos.\varphi^{2n-2})}.$$

SOLUTIO

Cum igitur elementum huius curvae sit

$$\partial s = \partial\varphi \sqrt{(aa \sin.\varphi^{2n-2} + bb \cos.\varphi^{2n-2})},$$

evidens est huic conditioni satisfieri, si elementa coordinatarum, quae primo sint X et Y, ita constituantur

$$\partial X = a\partial\varphi \sin.\varphi^{n-1} \quad \text{et} \quad \partial Y = b\partial\varphi \cos.\varphi^{n-1},$$

quandoquidem hinc manifesto fit

$$\partial X^2 + \partial Y^2 = \partial s^2.$$

Verum quia hae formulae paucissimis casibus exceptis non forent integrabiles, eae nostro instituto minus inserviunt; at vero ex iis alias coordinatas, quae sint x et y, formare licebit, ubi integratio certe succedet. Quodsi enim in genere statuamus

$$\partial x = \partial X \cos.\omega - \partial Y \sin.\omega \quad \text{et} \quad \partial y = \partial X \sin.\omega + \partial Y \cos.\omega,$$

hinc utique fiet

$$\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2 = \partial s^2.$$

Hae autem singulae partes revera integrationem admittent, si capiamus

$$\omega = (n + 2i + 1)\varphi;$$

quamobrem si loco ∂X et ∂Y valores assumtos restituamus, ambae coordinatae x et y ita algebraice exprimentur, ut sit

$$x = a\int \partial\varphi \sin.\varphi^{n-1} \cos.(n + 2i + 1)\varphi - b\int \partial\varphi \cos.\varphi^{n-1} \sin.(n + 2i + 1)\varphi$$

et

$$y = a\int \partial\varphi \sin.\varphi^{n-1} \sin.(n + 2i + 1)\varphi + b\int \partial\varphi \cos.\varphi^{n-1} \cos.(n + 2i + 1)\varphi,$$

ubi hae quatuor formulae integrales ope nostrorum theorematum algebraice exhiberi poterunt, ita ut, dum pro i omnes numeros integros positivos non excepta cyphra assumere licet, infinitae curvae algebraicae problemati satisfacientes assignari poterunt, quarum simplicissima sumendo $i = 0$ erit his formulis contenta

$$x = \frac{a}{n}\sin.\varphi^{n} \cos.n\varphi + \frac{b}{n}\cos.\varphi^{n} \cos.n\varphi$$

et

$$y = \frac{a}{n}\sin.\varphi^{n} \sin.n\varphi + \frac{b}{n}\cos.\varphi^{n} \sin.n\varphi,$$

quae ergo succincte ita referri possunt, ut sit

$$x = \frac{1}{n}\cos.n\varphi\,(a\sin.\varphi^{n} + b\cos.\varphi^{n})$$

et

$$y = \frac{1}{n}\sin.n\varphi\,(a\sin.\varphi^{n} + b\cos.\varphi^{n}).$$

Hinc patet fore

$$\frac{y}{x} = \text{tang.}\,n\varphi$$

et

$$\sqrt{(xx + yy)} = \frac{1}{n}(a\sin.\varphi^{n} + b\cos.\varphi^{n}).$$

Unde haud difficile erit pro quovis casu aequationem inter ipsas coordinatas x et y elicere.

COROLLARIUM 1

24. Elementum curvae

$$\partial s = \partial\varphi\sqrt{(aa\sin.\varphi^{2n-2} + bb\cos.\varphi^{2n-2})}$$

in plures alias formas notatu dignas transfundere licet. Veluti si ponatur $\sin.\varphi = v$, ob $\partial\varphi = \frac{\partial v}{\sqrt{(1-vv)}}$ erit

$$\partial s = \frac{\partial v}{\sqrt{(1-vv)}}\left(aav^{2n-2} + bb(1-vv)^{n-1}\right),$$

ubi operae pretium est notasse casu $n = 2$ fieri

$$\partial s = \frac{\partial v}{\sqrt{(1-vv)}}\sqrt{((aa-bb)vv+bb)},$$

qua forma elementum ellipseos exprimitur, ita ut ope huius problematis infinitae curvae algebraicae reperiri queant, quarum longitudinem per arcus ellipticos metiri liceat.[1])

COROLLARIUM 2

25. Pro alia transformatione ponamus

$$\sin.\varphi = \sqrt{\frac{1-v}{2}} \quad \text{et} \quad \cos.\varphi = \sqrt{\frac{1+v}{2}}$$

eritque $\partial\varphi = -\frac{\partial v}{2\sqrt{(1-vv)}}$ hincque ergo fiet

$$\partial s = -\frac{\partial v}{2\sqrt{(1-vv)}}\sqrt{\frac{aa(1-v)^{n-1} + bb(1+v)^{n-1}}{2^{n-1}}},$$

quae formula casu $n = 2$ abit in hanc

$$\partial s = -\frac{\partial v}{2\sqrt{(1-vv)}}\sqrt{\frac{aa+bb+(bb-aa)v}{2}},$$

qua itidem elementum ellipticum exprimitur.

COROLLARIUM 3

26. Quodsi porro ponamus $\text{tang}.\varphi = t$, erit

$$\sin.\varphi = \frac{t}{\sqrt{(1+tt)}} \quad \text{et} \quad \cos.\varphi = \frac{1}{\sqrt{(1+tt)}},$$

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1) Confer Commentationem 639 p. 436 laudatam. A. L.

tum vero $\partial\varphi = \frac{\partial t}{1+tt}$, quibus substitutis elementum curvae nostrae erit

$$\partial s = \frac{\partial t}{1+tt}\sqrt{\frac{aat^{2n-2}+bb}{(1+tt)^{n-1}}} \quad \text{sive} \quad \partial s = \partial t\sqrt{\frac{aat^{2n-2}+bb}{(1+tt)^{n+1}}},$$

unde sumendo $n=2$ iterum prodit elementum ellipticum

$$\partial s = \partial t\sqrt{\frac{aatt+bb}{(1+tt)^3}}.$$

SCHOLION

27. Ceterum quoniam in nostris theorematibus infiniti factores sunt indicati, per quos quaepiam formula differentialis multiplicata reddatur integrabilis, meminisse iuvabit in elementis calculi integralis methodum tradi solere, qua ex cognito uno tali factore innumerabiles alii reperiri possunt. Veluti si formula differentialis $v\partial x$ ducta in quantitatem p praebeat integrale $\int pv\partial x = q$, tum denotante Q functionem quamcunque ipsius q etiam multiplicator Qp formulam propositam $v\partial x$ reddet integrabilem. Cum enim sit $pv\partial x = \partial q$, erit $Qpv\partial x = Q\partial q$; unde quoties formula $\int Q\partial q$ est integrabilis, etiam factor ille Qp formulam propositam $v\partial x$ reddet integrabilem. Verum perspicuum est hunc casum toto coelo discrepare a formulis illis integralibus, quas in nostris theorematibus attulimus. Nam cum formula $\partial\varphi \sin.\varphi^{n-1}$ ducta in $\sin.(n+1)\varphi$ praebeat integrale $\frac{\sin.\varphi^n}{n}\sin.n\varphi$, hinc nemo certe secundum methodum memoratam reliquos multiplicatores idoneos, qui sunt

$$\sin.(n+3)\varphi, \quad \sin.(n+5)\varphi, \quad \sin.(n+7)\varphi \quad \text{etc.},$$

tum vero etiam

$$\cos.(n+1)\varphi, \quad \cos.(n+3)\varphi, \quad \cos.(n+5)\varphi \quad \text{etc.},$$

elicere valebit; quamobrem illa theoremata tanto magis omni attentione digna sunt censenda.

DE ITERATA INTEGRATIONE FORMULARUM INTEGRALIUM DUM ALIQUIS EXPONENS PRO VARIABILI ASSUMITUR

Conventui exhibita die 19. Augusti 1776

Commentatio 653 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 7 (1789), 1793, p. 64—82
Summarium ibidem p. 40

SUMMARIUM

Pour mieux saisir l'idée de l'Auteur, le but de ce Mémoire, et la méthode d'intégrer qui y est développée, il suffira de suivre M. EULER dans la solution du premier Problème, où il traite la formule la plus simple $\int x^{\theta-1}\partial x$, dont la valeur prise depuis le terme $x=0$ jusqu'au terme $x=1$, est $\frac{1}{\theta}$, de sorte que $\int x^{\theta-1}\partial x=\frac{1}{\theta}$. Pour la nouvelle intégration de cette formule, on suppose l'exposant θ variable, et après avoir multiplié par $\partial\theta$, on prend de part et d'autre l'intégrale de manière qu'elle évanouisse en mettant $\theta=\alpha$. De cette manière on a

$$\int\frac{\partial x}{x}\int\partial\theta x^{\theta}=\int\frac{\partial\theta}{\theta}.$$

Mais à cause de $\frac{\partial x}{x}$ constant et

$$\int\partial\theta x^{\theta}=\frac{x^{\theta}-\alpha^{\theta}}{lx},$$

on aura

$$\int\frac{\partial x}{x}\cdot\frac{x^{\theta}-\alpha^{\theta}}{lx}=l\frac{\theta}{\alpha}.$$

D'une manière semblable, M. EULER traite plusieurs autres formules plus compliquées dont les intégrales sont connues pour certains termes d'intégration déterminés.

PROBLEMA 1

1. *Cum sit*

$$\int x^{\theta-1}\partial x\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{\theta},$$

hanc formulam denuo integrare sumto exponente θ variabili.

SOLUTIO

Quoniam hic de integratione agitur, ut ea determinetur, integrale ita capi assumamus, ut evanescat certo casu, posito scilicet $\theta=\alpha$. Multiplicetur ergo utrimque per elementum $\partial\theta$ et integratione iuxta hanc legem instituta pro parte dextra habebimus

$$\int\frac{\partial\theta}{\theta}=l\theta-l\alpha=l\frac{\theta}{\alpha}.$$

At pro parte sinistra notum est hanc integrationem a signo summatorio $\int$ penitus non turbari, et quia iam sola littera θ pro variabili habetur, $\frac{\partial x}{x}$ vero ut constans spectatur, ob $x^{\theta-1}\partial x=\frac{\partial x}{x}x^{\theta}$ habebimus

$$\int x^{\theta}\partial\theta=\frac{x^{\theta}}{lx}-\frac{x^{\alpha}}{lx};$$

quo valore substituto membrum sinistrum erit

$$\int\frac{\partial x}{x}\cdot\frac{x^{\theta}-x^{\alpha}}{lx},$$

quamobrem ista integratio iterata nos perducit ad hanc aequationem [1])

$$\int\frac{(x^{\theta-1}-x^{\alpha-1})\partial x}{lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\frac{\theta}{\alpha}.$$

COROLLARIUM 1

2. Si eodem modo formula integralis

$$\int x^{n+\theta-1}\partial x\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{1}{n+\theta}$$

1) Confer Commentationem 464 (indicis ENESTROEMIANI): *Nova methodus quantitates integrales determinandi*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 66; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 421, imprimis p. 427. A. L.

denuo integretur sumto θ variabili, reperietur haec aequatio integrata

$$\int (x^{n+\theta-1} - x^{n+\alpha-1}) \frac{\partial x}{lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l\frac{n+\theta}{n+\alpha}.$$

At si θ negative capiatur, tum etiam α negative accipi debebit, unde aequatio denuo integrata haec prodibit

$$\int (x^{n-\theta-1} - x^{n-\alpha-1}) \frac{\partial x}{lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l\frac{n-\theta}{n-\alpha}.$$

COROLLARIUM 2

3. Hic igitur notentur istae integrationes, quas in parte sinistra institui oportet et quibus pro aliis formulis in posterum erit utendum, ubi semper assumamus integralia ita capi debere, ut evanescant posito $\theta = \alpha$. Primo scilicet erit

$$\int x^\theta \partial\theta = \frac{x^\theta - x^\alpha}{lx},$$

praeterea vero simili modo

$$\int x^{n+\theta} \partial\theta = \frac{x^{n+\theta} - x^{n+\alpha}}{lx}$$

atque hinc porro intelligitur fore

$$\int x^{n+\lambda\theta} \partial\theta = \frac{x^{n+\lambda\theta} - x^{n+\lambda\alpha}}{\lambda lx},$$

unde patet, si λ capiatur negative, fore

$$\int x^{n-\lambda\theta} \partial\theta = \frac{x^{n-\lambda\theta} - x^{n-\lambda\alpha}}{-\lambda lx}.$$

PROBLEMA 2

4. *Cum sit, uti iam saepius*[1]) *est ostensum,*

$$\int \frac{x^{\theta-1}\partial x}{1+x^\nu} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{\pi}{\nu \sin.\frac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare sumto exponente θ pro variabili.

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1) Confer Commentationes 60 et 254 (indicis ENESTROEMIANI): *De inventione integralium, si post integrationem variabili quantitati determinatus valor tribuatur,* Miscellanea Berolin. 7, 1743, p. 129, et *De expressione integralium per factores*, Novi comment. acad. sc. Petrop. 6 (1756/7), 1761, p. 115; *LEONHARDI EULERI Opera omnia,* series I, vol. 17, p. 35 et 233, imprimis p. 54 et 260. A. L.

SOLUTIO

Perpetuo hic ut hactenus integralia ita accipi statuamus, ut evanescant posito $\theta=\alpha$; quo observato pro parte dextra habebimus

$$\int\frac{\pi\partial\theta}{\nu\sin.\frac{\theta\pi}{\nu}},$$

quae formula posito $\frac{\theta\pi}{\nu}=\varphi$ abit in hanc $\int\frac{\partial\varphi}{\sin.\varphi}$, cuius integrale novimus esse $l\,\mathrm{tang}.\frac{1}{2}\varphi$; quamobrem adiecta debita constante pro hac parte habebimus

$$\int\frac{\pi\partial\theta}{\nu\sin.\frac{\theta\pi}{\nu}}=l\,\mathrm{tang}.\frac{\theta\pi}{2\nu}-l\,\mathrm{tang}.\frac{\alpha\pi}{2\nu}=l\frac{\mathrm{tang}.\frac{\theta\pi}{2\nu}}{\mathrm{tang}.\frac{\alpha\pi}{2\nu}}.$$

Pro parte autem sinistra, ubi solus factor $x^{\theta-1}$ est variabilis, erit

$$\int x^{\theta-1}\partial\theta=\frac{x^{\theta-1}-x^{\alpha-1}}{lx}.$$

Hoc igitur valore introducto formula nostra integralis denuo integrata erit

$$\int\frac{\partial x(x^{\theta-1}-x^{\alpha-1})}{(1+x^{\nu})lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\frac{\mathrm{tang}.\frac{\theta\pi}{2\nu}}{\mathrm{tang}.\frac{\alpha\pi}{2\nu}}.$$

COROLLARIUM

5. Quodsi ergo sumamus $\alpha=\frac{1}{2}\nu$, quoniam $\mathrm{tang}.\frac{\pi}{4}=1$, hoc casu ponendo potius $\nu=2\alpha$ habebimus hanc aequationem integralem satis memorabilem [1])

$$\int\frac{\partial x(x^{\theta-1}-x^{\alpha-1})}{(1+x^{2\alpha})lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\mathrm{tang}.\frac{\theta\pi}{4\alpha}.$$

1) Confer Commentationem 463 (indicis ENESTROEMIANI): *De valore formulae integralis* $\int\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}}\cdot\frac{dz}{z}(lz)^{\mu}$ *casu, quo post integrationem ponitur* $z=1$, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 30; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 384, imprimis p. 415. Confer ibidem p. 421 Commentationem 464 supra p. 459 laudatam, imprimis p. 440. A. L.

PROBLEMA 3

6. *Cum sit, uti iam satis constat*[1],

$$\int \frac{(x^{\theta-1}+x^{\nu-\theta-1})\partial x}{1+x^{\nu}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{\nu \sin. \dfrac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare per exponentem variabilem θ, ita ut integralia evanescant posito $\theta = \alpha$.

SOLUTIO

Multiplicando igitur per $\partial\theta$ et integrando pro parte dextra prorsus ut in praecedente problemate habebimus

$$l\frac{\text{tang.}\dfrac{\theta\pi}{2\nu}}{\text{tang.}\dfrac{\alpha\pi}{2\nu}}.$$

Pro parte autem sinistra, quia formula $\frac{\partial x}{1+x^{\nu}}$ est constans et exponens θ in duobus terminis occurrit, pro priore termino habebimus

$$\int x^{\theta-1}\partial\theta = \frac{x^{\theta-1}-x^{\alpha-1}}{lx},$$

pro altero vero termino ex § 3 habebimus

$$\int x^{\nu-\theta-1}\partial\theta = \frac{x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{lx};$$

quibus valoribus substitutis orietur ista nova integratio

$$\int \frac{\partial x}{lx}\cdot\frac{x^{\theta-1}-x^{\alpha-1}+x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{1+x^{\nu}} \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\text{tang.}\dfrac{\theta\pi}{2\nu}}{\text{tang.}\dfrac{\alpha\pi}{2\nu}}.$$

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1) Confer praeter Commentationes 60 et 463 modo laudatas etiam Commentationes 59 et 462 (indicis ENESTROEMIANI): *Theoremata circa reductionem formularum integralium ad quadraturam circuli*, Miscellanea Berolin. 7, 1743, p. 91, et *De valore formulae integralis* $\int \frac{z^{m-1}\pm z^{n-m-1}}{1\pm z^{n}}dz$ *casu, quo post integrationem ponitur* $z=1$, Novi comment. acad. sc. Petrop. **19** (1774), 1775, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 17, p. 1 et 358, imprimis p. 9 et 375. A. L.

COROLLARIUM 1

7. Ista aequatio aliquanto succinctius ita repraesentari potest

$$\int \frac{\partial x}{x\,lx} \cdot \frac{x^{\theta} - x^{\alpha} + x^{\nu-\alpha} - x^{\nu-\theta}}{1 + x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \frac{\text{tang.} \frac{\theta\pi}{2\nu}}{\text{tang.} \frac{\alpha\pi}{2\nu}};$$

ubi cum sit

$$x^{\nu-\alpha} - x^{\nu-\theta} = x^{\nu-\alpha-\theta}(x^{\theta} - x^{\alpha}),$$

ista aequatio ita commodius per factores repraesentari poterit

$$\int \frac{\partial x}{x\,lx} \cdot \frac{(x^{\theta} - x^{\alpha})(1 + x^{\nu-\alpha-\theta})}{1 + x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \frac{\text{tang.} \frac{\theta\pi}{2\nu}}{\text{tang.} \frac{\alpha\pi}{2\nu}}.$$

COROLLARIUM 2

8. Quodsi hic capiamus $\theta = \nu - \alpha$, ut fiat $x^{\nu-\alpha-\theta} = 1$, pro parte dextra erit $\text{tang.} \frac{(\nu - \alpha)\pi}{2\nu} = \text{cotang.} \frac{\alpha\pi}{2\nu}$, unde totum hoc membrum erit $2l \cot. \frac{\alpha\pi}{2\nu}$; quare cum pro parte sinistra factor $1 + x^{\nu-\alpha-\theta}$ evadat $= 2$, utrimque per 2 dividendo habebimus

$$\int \frac{\partial x}{x\,lx} \cdot \frac{x^{\nu-\alpha} - x^{\alpha}}{1 + x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \cot. \frac{\alpha\pi}{2\nu}.$$

COROLLARIUM 3

9. Quodsi sumamus $\nu = 2\alpha$, ut fiat $\text{tang.} \frac{\alpha\pi}{2\nu} = 1$, pro parte sinistra factor $1 + x^{\nu-\alpha-\theta}$ abit in $1 + x^{\alpha-\theta}$, dum prior factor $x^{\theta} - x^{\alpha}$ ita repraesentari potest $x^{\theta}(1 - x^{\alpha-\theta})$, unde amborum productum erit $x^{\theta}(1 - x^{2\alpha-2\theta})$; quamobrem integratio nostra ita se habebit

$$\int \frac{x^{\theta-1}\partial x}{lx} \cdot \frac{1 - x^{2\alpha-2\theta}}{1 + x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \text{ tang.} \frac{\theta\pi}{4\alpha}.$$

SCHOLION

10. Istae integrationes eo maiorem attentionem merentur, quod in iis tres exponentes α, θ, ν indefiniti occurrunt, quos singulos pro lubitu utcunque determinare licet, ita ut istae formulae multo latius pateant, quam eae, quas non ita pridem[1]) ex iisdem fundamentis derivavi.

PROBLEMA 4

11. *Cum sit, uti iam abunde est demonstratum*[2]),

$$\int \frac{x^{\theta-1}-x^{\nu-\theta-1}}{1-x^{\nu}}\,\partial x \begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = \frac{\pi}{\nu\,\text{tang.}\,\frac{\theta\pi}{\nu}},$$

hanc formulam denuo integrare sumto exponente θ variabili, ita ut integralia evanescant posito $\theta=\alpha$.

SOLUTIO

Quodsi ergo hic per $\partial\theta$ multiplicemus, pro parte dextra habebimus

$$\frac{\pi\partial\theta}{\nu\,\text{tang.}\,\frac{\theta\pi}{\nu}},$$

quae formula posito $\frac{\pi\theta}{\nu}=\varphi$ abit in

$$\frac{\partial\varphi}{\text{tang.}\,\varphi}=\frac{\partial\varphi\cos.\varphi}{\sin.\varphi},$$

cuius integrale manifesto est $l\sin.\varphi$; quamobrem constanti debita adiecta pro parte dextra habebimus

$$l\sin.\frac{\theta\pi}{\nu}-l\sin.\frac{\alpha\pi}{\nu}=l\,\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}.$$

Pro parte autem sinistra, quae ita repraesentetur

$$\int\frac{\partial x}{x}\cdot\frac{x^{\theta}-x^{\nu-\theta}}{1-x^{\nu}},$$

1) Confer Commentationes 463 et 464 modo laudatas. A. L.

2) Confer Commentationes 59, 60, 462 et 463 supra laudatas. A. L.

habebimus

$$\int x^\theta \partial\theta = \frac{x^\theta - x^\alpha}{lx} \quad \text{et} \quad \int x^{\nu-\theta} \partial\theta = \frac{x^{\nu-\alpha} - x^{\nu-\theta}}{lx},$$

quibus valoribus substitutis orietur sequens aequatio integrata

$$\int \frac{\partial x}{xlx} \cdot \frac{x^\theta - x^\alpha - x^{\nu-\alpha} + x^{\nu-\theta}}{1 - x^\nu} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \frac{\sin. \frac{\theta\pi}{\nu}}{\sin. \frac{\alpha\pi}{\nu}},$$

ubi iterum tres exponentes indefiniti occurrunt α, θ, ν.

COROLLARIUM 1

12. Cum sit, uti iam ante observavimus,

$$x^{\nu-\alpha} - x^{\nu-\theta} = x^{\nu-\alpha-\theta}(x^\theta - x^\alpha),$$

formula nostra commodius ita per factores exprimi poterit

$$\int \frac{\partial x}{xlx} \cdot \frac{(x^\theta - x^\alpha)(1 - x^{\nu-\alpha-\theta})}{1 - x^\nu} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \frac{\sin. \frac{\theta\pi}{\nu}}{\sin. \frac{\alpha\pi}{\nu}};$$

ubi si sumeremus $\nu = \alpha + \theta$, membrum sinistrum evanesceret, dextrum autem manifesto quoque evanesceret.

COROLLARIUM 2

13. Quodsi autem hic sumamus $\nu = 2\alpha$, pro dextra foret $\sin. \frac{\alpha\pi}{\nu} = 1$, unde hoc casu formula nostra integralis erit

$$\int \frac{\partial x}{xlx} \cdot \frac{(x^\theta - x^\alpha)(1 - x^{\alpha-\theta})}{1 - x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \sin. \frac{\theta\pi}{2\alpha},$$

quae forma evidenter in hanc contrahitur

$$\int \frac{x^{\theta-1} \partial x}{lx} \cdot \frac{(1 - x^{\alpha-\theta})^2}{1 - x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \sin. \frac{\theta\pi}{2\alpha}.$$

SCHOLION

14. Has igitur egregias integrationes deduximus ex formulis integralibus iam pridem erutis, quatenus in iis exponentes indefiniti occurrunt; quodsi ergo aliae huiusmodi formulae integrales insuper innotescerent, eas simili

modo tractare liceret; verum hactenus nullae tales formulae sunt inventae, quae ad hunc scopum accommodari possunt, quam ob causam integrationes hic exhibitae summa attentione Geometrarum dignae sunt existimandae.

ADDITAMENTUM

15. Cum nuper[1]) ostendissem huius formulae integralis

$$\int \frac{x^{a-1}\partial x}{lx} \cdot \frac{(1-x^b)(1-x^c)}{1-x^n}$$

a termino $x=0$ ad terminum $x=1$ extensae valorem ita exprimi, ut sit $l\,\frac{P}{Q}$ existente

$$P=\int \frac{x^{a+b-1}\partial x}{(1-x^n)^{1-\frac{c}{n}}} \quad \text{et} \quad Q=\int \frac{x^{a-1}\partial x}{(1-x^n)^{1-\frac{c}{n}}},$$

quae integralia denuo ab $x=0$ ad $x=1$ sunt extendenda, manifestum est in hac forma generali plerasque integrationes supra inventas contineri; quamobrem cum illis casibus valores integralium absolute exprimantur, operae pretium erit istam formam generalem ad illos casus applicare, ut relatio inter binas formulas integrales P et Q inde innotescat. Problema quidem primum et secundum huc plane non pertinent. Ex problemate igitur tertio et quarto eos perscrutemur casus, quos ad formam nostram generalem revocare licet.

EVOLUTIO FORMULAE INTEGRALIS SUPRA § 8 INVENTAE

$$\int \frac{\partial x}{xlx} \cdot \frac{x^{\nu-\alpha}-x^\alpha}{1+x^\nu} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l \cot. \frac{\alpha\pi}{2\nu}$$

16. Quoniam hic denominator est $1+x^\nu$, ut is ad formam generalem reducatur, multiplicetur fractio supra et infra per $1-x^\nu$ et formula ista integralis hanc induet formam

$$\int \frac{\partial x}{xlx} \cdot \frac{(x^{\nu-\alpha}-x^\alpha)(1-x^\nu)}{1-x^{2\nu}}.$$

Hic ante omnia dispiciendum est, uter exponentium $\nu-\alpha$ et α sit maior, unde duos casus evolvi conveniet, prouti fuerit vel $\nu-\alpha<\alpha$, hoc est $\nu<2\alpha$, vel $\nu-\alpha>\alpha$, hoc est $\nu>2\alpha$.

1) Scilicet in Commentatione 500 huius voluminis p. 51. A. L.

17. Sit igitur primo $\nu < 2\alpha$ seu $\alpha > \frac{1}{2}\nu$ atque formula integralis ita repraesentari poterit

$$\int \frac{x^{\nu-\alpha-1}\partial x}{lx} \cdot \frac{(1-x^{2\alpha-\nu})(1-x^\nu)}{1-x^{2\nu}}.$$

Hinc iam comparatione cum formula generali instituta manifesto habebimus $a = \nu - \alpha$, $b = 2\alpha - \nu$ et $c = \nu$, denique $n = 2\nu$, ex quibus valoribus formabuntur sequentes formulae

$$P = \int \frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}} \quad \text{et} \quad Q = \int \frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}.$$

Ponere etiam potuissemus $b = \nu$ et $c = 2\alpha - \nu$ manentibus $a = \nu - \alpha$ et $n = 2\nu$ hincque prodiissent valores

$$P = \int \frac{x^{2\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}};$$

utrimque autem erit $l\frac{P}{Q} = l \cot. \frac{\alpha\pi}{2\nu}$.

18. Hinc igitur duas nanciscimur integrationes notatu dignissimas. Cum enim sit $\frac{P}{Q} = \cot. \frac{\alpha\pi}{2\nu}$, hae duae integrationes ita se habebunt

$$\text{I.} \quad \int \frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}} : \int \frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}} = \cot. \frac{\alpha\pi}{2\nu},$$

$$\text{II.} \quad \int \frac{x^{2\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}} : \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{3\nu-2\alpha}{2\nu}}} = \cot. \frac{\alpha\pi}{2\nu}.$$

19. Sin autem fuerit $\nu > 2\alpha$, ipsa formula generalis mutatis signis ita debet repraesentari

$$\int \frac{\partial x}{x\,lx} \cdot \frac{(x^\alpha - x^{\nu-\alpha})(1-x^\nu)}{1-x^{2\nu}} = l \operatorname{tang.} \frac{\alpha\pi}{2\nu},$$

cui aequationi nunc induamus hanc formam

$$\int \frac{x^{\alpha-1}\partial x}{lx} \cdot \frac{(1-x^{\nu-2\alpha})(1-x^\nu)}{1-x^{2\nu}},$$

unde iam manifesto habemus $a=\alpha$, $b=\nu-2\alpha$, $c=\nu$ atque $n=2\nu$, unde deducuntur isti valores

$$P=\int\frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}\quad\text{et}\quad Q=\int\frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}.$$

Sin autem sumamus $c=\nu-2\alpha$ et $b=\nu$ manente $a=\alpha$ et $n=2\nu$, reperietur

$$P=\int\frac{x^{\alpha+\nu-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}\quad\text{et}\quad Q=\int\frac{x^{\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}.$$

20. Cum nunc utrimque sit $l\frac{P}{Q}=l\operatorname{tang.}\frac{\alpha\pi}{2\nu}$ ideoque $\frac{P}{Q}=\operatorname{tang.}\frac{\alpha\pi}{2\nu}$, hinc adipiscimur iterum has duas integrationes

$$\text{III.}\quad\int\frac{x^{\nu-\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}:\int\frac{x^{\alpha-1}\partial x}{\sqrt{(1-x^{2\nu})}}=\operatorname{tang.}\frac{\alpha\pi}{2\nu},$$

quae quidem convenit cum priore antecedentium, siquidem formulae P et Q tantum inter se permutantur; altera vero integratio est nova, scilicet

$$\text{IV.}\quad\int\frac{x^{\alpha+\nu-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}:\int\frac{x^{\alpha-1}\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}=\operatorname{tang.}\frac{\alpha\pi}{2\nu}.$$

EVOLUTIO FORMULAE INTEGRALIS § 9 ALLATAE

$$\int\frac{x^{\theta-1}\partial x}{lx}\cdot\frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=l\operatorname{tang.}\frac{\theta\pi}{4\alpha}$$

21. Quo haec expressio ad formam praescriptam reducatur, multiplicetur supra et infra per $1-x^{2\alpha}$, ut habeamus hanc formam

$$\int\frac{x^{\theta-1}\partial x}{lx}\cdot\frac{(1-x^{2\alpha-2\theta})(1-x^{2\alpha})}{1-x^{4\alpha}}=l\operatorname{tang.}\frac{\theta\pi}{4\alpha},$$

quae sponte ad formam generalem revocatur sumendo $a=\theta$, $b=2\alpha-2\theta$, $c=2\alpha$ et $n=4\alpha$, si modo fuerit $\alpha>\theta$. Si enim fuerit $\theta>\alpha$, alio modo comparatio institui debet, uti deinceps videbimus. Ex his autem valoribus conficietur

$$P=\int\frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}}\quad\text{et}\quad Q=\int\frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}},$$

unde ergo deducitur

$$\text{V.}\quad \int \frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} : \int \frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} = \text{tang.}\,\frac{\theta\pi}{4\alpha}.$$

22. Possumus etiam valores litterarum b et c inter se permutare, ut sit $b = 2\alpha$ et $c = 2\alpha - 2\theta$ manentibus $a = \theta$ et $n = 4\alpha$; tum autem fiet

$$P = \int \frac{x^{2\alpha+\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} \quad \text{et} \quad Q = \int \frac{x^{\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}}$$

hincque deducitur reductio

$$\text{VI.}\quad \int \frac{x^{2\alpha+\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} : \int \frac{x^{\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} = \text{tang.}\,\frac{\theta\pi}{4\alpha},$$

quae autem aeque ac praecedens locum non habet, nisi sit $\alpha > \theta$.

23. Quodsi autem θ superet α, aequationem nostram in aliam formam transfundi oportet signa utrimque mutando, unde prodibit

$$\int \frac{x^{2\alpha-\theta-1}\partial x}{lx} \cdot \frac{(1-x^{2\theta-2\alpha})(1-x^{2\alpha})}{1-x^{4\alpha}} = l\,\text{cot.}\,\frac{\theta\pi}{4\alpha}.$$

Hic iam iterum duplex comparatio institui potest; primo scilicet sumamus $a = 2\alpha - \theta$, $b = 2\theta - 2\alpha$, $c = 2\alpha$ et $n = 4\alpha$, unde formamus

$$P = \int \frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} \quad \text{et} \quad Q = \int \frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}},$$

hincque oritur septima relatio haec

$$\text{VII.}\quad \int \frac{x^{\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} : \int \frac{x^{2\alpha-\theta-1}\partial x}{\sqrt{(1-x^{4\alpha})}} = \text{cot.}\,\frac{\theta\pi}{4\alpha},$$

quae manifesto cum quinta congruit.

24. Nova autem reductio obtinebitur, si statuamus $b = 2\alpha$ et $c = 2\theta - 2\alpha$ manentibus $a = 2\alpha - \theta$ et $n = 4\alpha$; tum igitur erit

$$P = \int \frac{x^{4\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} \quad \text{et} \quad Q = \int \frac{x^{2\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}}.$$

Hinc vero colligitur reductio octava

$$\text{VIII.} \quad \int \frac{x^{4\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} : \int \frac{x^{2\alpha-\theta-1}\partial x}{(1-x^{4\alpha})^{\frac{3\alpha-\theta}{2\alpha}}} = \cot. \frac{\theta\pi}{4\alpha}.$$

25. Hic autem probe notandum est quaternas posteriores reductiones ex quatuor prioribus oriri, si in istis loco α scribatur θ, at ν loco 2α, ita ut quatuor posteriores reductiones iam in prioribus contineantur; quamobrem sive quatuor priores sive posteriores penitus omittere licebit, ita ut nobis tantum quatuor relinquantur, inter quas porro, quoniam tertia non discrepat a prima, tantum tres supererunt huiusmodi reductiones, quae quidem ex problemate tertio sunt natae.

EVOLUTIO FORMULAE INTEGRALIS § 12 ALLATAE

$$\int \frac{\partial x}{x\,lx} \cdot \frac{(x^{\theta}-x^{\alpha})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l\,\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}$$

26. Ista expressio iam congruit cum forma nostra generali neque idcirco ulteriori transformatione indiget. Hic quidem duo casus essent distinguendi, prouti fuerit vel $\theta > \alpha$ vel $\theta < \alpha$; verum hac etiam distinctione carere possumus, propterea quod binae litterae α et θ inter se sunt permutabiles; iis enim permutatis signa utrimque invertuntur. Hanc ob causam, quoscunque valores habuerint ambae litterae α et θ, minorem semper littera θ, maiorem vero littera α designare licebit, unde aequatio nostra ita repraesentabitur

$$\int \frac{x^{\theta-1}\partial x}{lx} \cdot \frac{(1-x^{\alpha-\theta})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} = l\,\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}.$$

27. Nihilo vero minus duo casus distinguendi etiam hic occurrunt, prouti fuerit vel $\nu > \alpha + \theta$ vel $\nu < \alpha + \theta$. Sit igitur primo $\nu > \alpha + \theta$ et forma exposita manebit invariata, quae denuo duplicem comparationem cum generali admittit. Primo igitur statuamus $a = \theta$, $b = \alpha - \theta$, $c = \nu - \alpha - \theta$ et $n = \nu$, qui valores nobis suppeditant

$$P = \int \frac{x^{\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} \quad \text{et} \quad Q = \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}},$$

sicque ex hac evolutione habebimus sequentem reductionem

$$\text{I.}\quad \int \frac{x^{\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} = \frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}.$$

28. Secunda nascetur reductio permutandis litteris b et c, ita ut sit $a=\theta$, $b=\nu-\alpha-\theta$, $c=\alpha-\theta$ et $n=\nu$, unde formantur hae formulae

$$P=\int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} \quad \text{et} \quad Q=\int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}};$$

quare secunda reductio hinc orta erit

$$\text{II.}\quad \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} = \frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}},$$

quae duae reductiones postulant, ut sit $\nu>\alpha+\theta$.

29. Sin autem fuerit $\nu<\alpha+\theta$, ipsa aequationis forma hoc modo immutari debebit

$$\int \frac{x^{\nu-\alpha-1}\partial x}{lx}\cdot\frac{(1-x^{\alpha-\theta})(1-x^{\alpha+\theta-\nu})}{1-x^{\nu}} = l\frac{\sin.\frac{\alpha\pi}{\nu}}{\sin.\frac{\theta\pi}{\nu}},$$

ubi iterum gemina comparatio institui potest. Sit igitur primo $a=\nu-\alpha$, $b=\alpha-\theta$, $c=\alpha+\theta-\nu$ et $n=\nu$, unde oriuntur hae formulae

$$P=\int \frac{x^{\nu-\theta-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}} \quad \text{et} \quad Q=\int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}}.$$

Hinc igitur concluditur tertia reductio

$$\text{III.}\quad \int \frac{x^{\nu-\theta-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}} : \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{2\nu-\alpha-\theta}{\nu}}} = \frac{\sin.\frac{\alpha\pi}{\nu}}{\sin.\frac{\theta\pi}{\nu}}.$$

30. Denique statuamus $a=\nu-\alpha$, $b=\alpha+\theta-\nu$, $c=\alpha-\theta$ et $n=\nu$ et formulae hinc sequentes nascentur

$$P=\int \frac{x^{\theta-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} \quad \text{et} \quad Q=\int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}},$$

ita ut quarta hinc oriatur reductio

$$\text{IV.}\quad \int \frac{x^{\theta-1}\partial x}{(1-x^\nu)^{\frac{\nu-\alpha+\theta}{\nu}}} : \int \frac{x^{\nu-\alpha-1}\partial x}{(1-x^\nu)^{\frac{\nu-\alpha+\theta}{\nu}}} = \frac{\sin.\frac{\alpha\pi}{\nu}}{\sin.\frac{\theta\pi}{\nu}}.$$

31. Quatuor igitur hic nacti sumus formularum integralium paria, quae eandem inter se tenent rationem ac sinus duorum angulorum, dum evolutiones praecedentes tantum tria huiusmodi paria praebuerant, quarum ratio $P:Q$ tangenti cuiuspiam anguli aequatur, ubi quidem evidens est secundam et quartam inter se convenire. Cum igitur huiusmodi reductiones altioris sint indaginis ac sine dubio insignem usum habere queant, operae pretium erit eas clarius ob oculos exponere.

PROBLEMA

32. *Invenire binas formulas integrales P et Q ab $x=0$ ad $x=1$ extensas, ut fiat*

$$\frac{P}{Q} = \text{tang.}\frac{m\pi}{2n}.$$

SOLUTIO

Triplici igitur modo hoc fieri potest secundum evolutionem primam supra institutam.

I. Ex prima enim reductione, cum sit $\cot.\frac{\alpha\pi}{2\nu} = \text{tang.}\frac{(\nu-\alpha)\pi}{2\nu}$, fiet $\nu-\alpha=m$ et $\nu=n$, ita ut sit $\alpha=n-m$. Hinc igitur erit

$$P = \int \frac{x^{n-m-1}\partial x}{\sqrt{(1-x^{2n})}} \quad \text{et} \quad Q = \int \frac{x^{m-1}\partial x}{\sqrt{(1-x^{2n})}},$$

quae ergo est solutio prima.

II. Secunda reductio supra allata erat $\frac{P}{Q} = \cot.\frac{\alpha\pi}{2\nu} = \text{tang.}\frac{(\nu-\alpha)\pi}{2\nu}$, ubi ergo iterum est $\alpha=n-m$ et $\nu=n$, sicque secunda solutio huius problematis constabit his formulis

$$P = \int \frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}.$$

Hae autem formulae tantum valent, quando fuerit $m < \frac{1}{2}n$ ideoque ipse angulus $\frac{m\pi}{2n}$ minor semirecto.

III. Quoniam tertia reductio ibi allata cum prima convenit, ex quarta, ubi erat $\frac{P}{Q} = \text{tang.}\frac{\alpha\pi}{2\nu}$ ideoque pro nostro casu $\alpha = m$ et $\nu = n$, tertia solutio ita se habebit

$$P = \int \frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}};$$

qui valores quoniam a praecedentibus non sint diversi, duas tantum adipiscimur solutiones nostri problematis, quarum secunda limitatione quadam indiget, scilicet $m < \frac{1}{2}n$, prior vero ad omnes angulos recto non maiores patet. Hae ergo duae solutiones ita repraesententur

$$\text{I.} \quad P = \int \frac{x^{n-m-1}\partial x}{\sqrt{(1-x^{2n})}}, \quad Q = \int \frac{x^{m-1}\partial x}{\sqrt{(1-x^{2n})}},$$

$$\text{II.} \quad P = \int \frac{x^{m+n-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}, \quad Q = \int \frac{x^{m-1}\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}};$$

ex utraque igitur erit $\frac{P}{Q} = \text{tang.}\frac{m\pi}{2n}$.

PROBLEMA

33. *Invenire binas formulas integrales P et Q, ut fiat*

$$\frac{P}{Q} = \frac{\sin.\frac{p\pi}{2n}}{\sin.\frac{q\pi}{2n}},$$

siquidem ambo illa integralia ab $x = 0$ ad $x = 1$ extendantur.

SOLUTIO

Ad hanc igitur formam transferamus quatuor illas reductiones in evolutione tertia traditas, et cum pro prima et secunda esset

$$\frac{P}{Q} = \frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}},$$

pro forma hic praescripta erit $\theta = p$, $\alpha = q$ et $\nu = 2n$; quamobrem hinc nanciscimur duas sequentes solutiones

$$\text{I. } P = \int \frac{x^{q-1}\partial x}{(1-x^{2n})^{\frac{p+q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{p+q}{2n}}},$$

$$\text{II. } P = \int \frac{x^{2n-q-1}\partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}}.$$

Tertia vero et quarta reductio habebant

$$\frac{P}{Q} = \frac{\sin.\frac{\alpha\pi}{\nu}}{\sin.\frac{\theta\pi}{\nu}},$$

pro qua igitur erit $\alpha = p$, $\theta = q$, $\nu = 2n$, unde ambae solutiones sequentes deducuntur

$$\text{III. } P = \int \frac{x^{2n-q-1}\partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}},$$

$$\text{IV. } P = \int \frac{x^{q-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}}.$$

Hinc igitur patet quadruplici modo fieri posse

$$\frac{P}{Q} = \frac{\sin.\frac{p\pi}{2n}}{\sin.\frac{q\pi}{2n}}.$$

COROLLARIUM 1

34. Si assumamus $q = n$, ut fiat $\sin.\frac{q\pi}{2n} = 1$ ideoque prodire debeat $\frac{P}{Q} = \sin.\frac{p\pi}{2n}$, pro hoc casu quatuor inventae solutiones dabunt

$$\text{I. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}},$$

$$\text{II. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}} \quad \text{et} \quad Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}},$$

$$\text{III.}\quad P=\int\frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}}\quad\text{et}\quad Q=\int\frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

$$\text{IV.}\quad P=\int\frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}}\quad\text{et}\quad Q=\int\frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}}.$$

ubi ergo solutio prima cum secunda et tertia cum quarta convenit.

COROLLARIUM 2

35. Sumamus nunc esse $q=n-p$, ut fiat $\sin.\frac{q\pi}{2n}=\cos.\frac{p\pi}{2n}$ ideoque prodire debeat $\frac{P}{Q}=\operatorname{tang.}\frac{p\pi}{2n}$. Pro hoc ergo casu quatuor solutiones inventae evadent

$$\text{I.}\quad P=\int\frac{x^{n-p-1}\partial x}{\sqrt{(1-x^{2n})}}\quad\text{et}\quad Q=\int\frac{x^{p-1}\partial x}{\sqrt{(1-x^{2n})}},$$

$$\text{II.}\quad P=\int\frac{x^{n+p-1}\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}}\quad\text{et}\quad Q=\int\frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}},$$

$$\text{III.}\quad P=\int\frac{x^{n+p-1}\partial x}{(1-x^{2n})^{\frac{3}{2}}}\quad\text{et}\quad Q=\int\frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3}{2}}},$$

$$\text{IV.}\quad P=\int\frac{x^{n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}}\quad\text{et}\quad Q=\int\frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}}$$

hincque erit

$$\frac{P}{Q}=\operatorname{tang.}\frac{p\pi}{2n},$$

ubi prima et secunda forma cum iis, quas in praecedente problemate invenimus, prorsus conveniunt; tertia autem forma ob $(1-x^{2n})^{\frac{3}{2}}$ fit incongrua, quia inde P et Q in infinitum excrescerent; quarta autem novam formam dare videtur.